Marx's Reproduction Schema and the Multisectoral Foundations of the Domar Growth Model

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1. Introduction

One of the most important contributions of Marx's economics has been the reproduction schema developed in *Capital*,Volume 2. These have been widely acclaimed as providing the forerunner to modern growth theory, and in particular to the Harrod-Domar growth model. Like Harrod and Domar, Marx demonstrates the (unlikely) conditions under which a capitalist economy can follow a balanced growth path.

Despite the similarities, however, the Harrod-Domar model is usually presented as a one-good framework, in contrast to Marx's multisectoral reproduction schema. Lianos (1979), for example, examines the relationship between Domar's version of the model and just one of the departments from Marx's schema. Similarly, Samuelson and Wolfson (1986) use an aggregate, implicitly one-good, production function to examine Marxian growth models. Moreover, in relation to the post Harrod-Domar growth literature, Geoffrey Hodgson has pointed out that 'Versions of aggregate production functions abound and are central to recent fashionable developments such as real business cycle theory and endogenous growth theory' (Hodgson, 1997, p. 104).

The contribution of this paper will be to derive the particular model developed by Domar (1957) from microfoundations that are consistent with Marx's multisectoral schema. Two main steps are required for this derivation. First, a role can be identified for the Keynesian multiplier in Marx's reproduction schema, thereby providing an interface with the Domar model. Under Marx's assumption in *Capital*, Volume 2, that prices are equivalent to values, the reproduction schema are interpreted as a Leontief input-output framework from which a Keynesian multiplier relationship can be established.

Second, by applying the so-called ‘new solution’ or ‘new interpretation’ of the transformation problem (Foley, 1982; Lipietz, 1982), Marx’s economic categories can be expressed in macroeconomic terms that are valid when prices diverge from values. Using the ‘value of money’ as a way of translating between money and labour categories, a multisectoral multiplier can be developed that is nested in the reproduction schema but can also be re-expressed in aggregate terms. This aggregation procedure allows a transition between the reproduction schema and the aggregate Domar model. In contrast to much of neoclassical growth theory, Marx’s reproduction schema can be used to derive a model of economic growth that is derived from multisectoral foundations.

Section 2 shows how an aggregate multiplier framework can be established in Marx’s reproduction schema. In Section 3 the ‘new interpretation’ is used to develop an aggregate multiplier relationship that can be nested in the Domar growth model.

2. The Reproduction Schema

The most developed of Marx’s expanded reproduction schema in *Capital,* Volume 2 is the ‘First Example’, referred to as ‘schema (B)’ in Chapter 21, Section 3 (Marx 1978, pp. 586-589). Table 1 shows this two-department model, with Department I producing investment goods and Department II consumption goods. Numerical elements of the table are made of up of constant capital (C_i), variable capital (V_i) and surplus value (S_i). Throughout the schema a constant rate of surplus value of 100 per cent is assumed together with 4:1 ratio of constant to variable capital in Department I and a 2:1 ratio in Department II. Constant capital inputs are non-durable, used up during a single period of production, and £1 of output is assumed equal to a unit of labour.

[Table 1 here]

Key to this economy's capacity to expand is the production of sufficient surplus value to invest in additional units of capital. Marx assumes that a half of surplus value in Department I is invested in this way. For Year 1 this means that 500 of the total 1000 units of surplus value produced in Department I are directed to 400 units of new constant capital and 100 units of new variable capital. In Year 2 constant capital expands from 4000 to 4400 units, and variable capital from 1000 to 1100 units, maintaining the 4:1 ratio between constant and variable capital. A new position of balance is established by also maintaining department II at its original 2:1 ratio.

Examination of the elements of Table 1 shows that from Year 3 onwards each department, and hence the economy as a whole, expands at a balanced growth rate of 10 per cent. Total output of 11858 in Year 4, for example, represents a 10 per cent increase upon the 10780 produced in Year 3. In the analysis that follows the conditions required to establish this balanced growth path will be explored by relating the multisectoral reproduction schema to the Domar model of economic growth.

Following Trigg (2001a) the reproduction schema can be re-cast in the form of a closed Leontief input-output model. In order to examine the demand side of the economy, a three sector dissaggregation is introduced, along the lines suggested by Kalecki (1968, p. 459). In this approach a distinction is made between the production of goods for consumption by capitalists (Department 2) and wage goods (Department 3). More detail of the structure of demand is also provided by explicitly distributing surplus value between capitalists' consumption (u_i), and new investment in constant

(dC_i) and variable capital (dV_i). Table 2 provides a numerical and algebraic representation of this model.

[Table 2 here]

In contrast to Table 1, the inputs of Table 2 are read column-wise. In the first column, for example, Department 1 uses 4000 units of constant capital, 1000 units of variable capital, from which 1000 units of surplus value are extracted. (In Table 1 these elements are represented in the first row of Marx's original layout). Reading along the first row of Table 2, Department 1 produces 4000 units of constant capital for its own use, 550 that are directed to Department 2, 950 directed to Department 3, and 500 directed to new constant capital in the next period of production.

Examination of the outputs of each department (X_i) in Table 2 shows that the outputs of Departments 2 and 3 add up to the original total for Department II in Table 1 (i.e. 1100+1900=3000). Hence the total output of the economy (9000) is the same as in the original reproduction scheme. Each of these two new departments have the same 2:1 ratio of constant to variable capital as the original Department II and a 100 per cent rate of surplus value.

The additions to constant and variable capital, to be used in the next period, are also the same as in Table 2; Department 1 producing 500 additional units of constant capital, and Department 2 producing 150 units of additional variable capital. The 1100 output of the new Department 3 is directed to capitalists' consumption.

This three-department reproduction scheme can be expressed algebraically by defining technical coefficients $a_{ij} = T_{ij} / X_j$ that specify the ratio between total flows of materials of production (T_{ij}), from department i to department j, to gross output

(X_j) of department j. Ratios to gross output of the total number of labour units employed in each sector (L_j) are represented by labour coefficients $l_j = L_j / X_j$; and consumption coefficients $h_i = C_i / L$ are ratios of total consumption of each good (C_i) to the total volume of labour units (L).[1]

The terms in this closed input-output model can be collected in block matrix form by writing:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ h_3 \end{bmatrix} \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} dC \\ u \\ dV \end{bmatrix} \qquad (1)$$

By specifying X as the column vector of gross outputs for each sector, A the square matrix of interindustry technical coefficients, h the column vector of worker consumption coefficients, l the row vector of labour coefficients, and F as the column vector representing final demand:

$$X = AX + hlX + F \qquad (2)$$

With final outputs defined as $Y = (I - A)X$ it follows that (2) can be re-expressed as:

$$Y = hvY + F \qquad (3)$$

where $v = l(I - A)^{-1}$ is Pasinetti's (1981) row vector of vertically integrated labour coefficients.

Pre-multiplication of (3) by the row vector v yields

$$vY = vhvY + vF \qquad (4)$$

and hence:

$$vY = \frac{1}{1-vh} vF \qquad (5)$$

Under Marx's assumption, in *Capital* Volume 2, that prices and values are proportional, and hence £1 of output is equal to an hour of labour time, this equation captures both an income and multiplier relationship. This proportionality is embodied in the identity $v = i'$ [2] such that the total employment of labour units vY is equal to total money net income y. Similarly vF, the total number of labour units required to produce final demand, is equal total money final demand (f) The expression $1/1 - vh$ is an income/employment multiplier with a particularly interesting denominator.

As argued in Trigg (2001a), the term vh can be interpreted to represent the value of labour power: the labour embodied (v) in the bundle consumed by workers per unit of labour (h). The denominator can therefore be expressed as e, the corresponding per capita share of surplus value extracted from each unit of labour. It follows that (5) can be expressed as a macro income multiplier:

$$y = \frac{1}{e} f \qquad (6)$$

This expression can be related back to Marx's numerical example in Tables 1 and 2. Since $e = 1 - vh = 1 - h_3 = 1 - \frac{1}{2} = \frac{1}{2}$ it follows that the multiplier $(1/e)$ takes a value of 2. This multiplier generates a total net income of $y = L = 3500$ from final demand $f = dC + dV + u = 1750$. In terms of Year 1 in Table 1 the net income is equal total variable capital added to total surplus value. Final demand is equal to the amount of surplus value produced in Year 1. The economy produces a volume of surplus value (1750) that is available for capitalists' consumption and investment in new capital; and the realisation of this surplus value is made possible by this final demand taking place. The multiplier determines the amount of net income generated from final demand.

The problem with the multiplier in equation (6) is that, although derived from multisectoral foundations, its generality is limited by the assumption that prices are equivalent to values. Marx maintains this assumption in *Capital* Volume 2 despite using examples in which the organic composition varies between sectors, as shown in Table 1. There are, of course, methodological reasons for this assumption, with Marx building up the level of complexity throughout the volumes of *Capital*. But in order to develop a more realistic and general model of economic growth this assumption has to be relaxed.

In the next part of the paper a way of allowing price-value deviations is suggested by introducing the 'new interpretation' approach associated with Foley (1982) and Lipietz (1982).

3. Domar and the 'New Interpretation'

Before introducing the Domar model a more general aggregate multiplier relationship is derived from the 'new interpretation'.

The New Interpretation

The main contribution of the 'new interpretation' is to provide an alternative definition of the value of labour power. Instead of measuring the value of labour power (VLP) as the labour embodied in commodities consumed by workers, the money wage rate is transformed into units of labour by the expression

$$VLP = \lambda_m w \qquad (7)$$

where w is the money wage rate and λ_m is the value of money:[3]

$$\lambda_m = \frac{lX}{y} .$$

The value of money represents the ratio of total labour time to money net output; its inverse commonly referred to as the 'monetary expression of labour time'. On this basis for Foley (1982, 37) 'a consistent interpretation of the labour theory of value is constructed in which surplus value is conserved in the transformation from labour values to prices, but in which the value of labour power is not in general equal to the labour value of workers' consumption'.

As formally demonstrated by Mohun (1994), this definition of the VLP enables an aggregate conservation of the relationship between surplus value and profits. Money profits and surplus value, measured in units of labour time, are directly proportional for any price vector. It is immaterial whether organic compositions vary between sectors, and prices deviate from values. Under the new interpretation the so called ‘transformation problem’ is abolished (Foley 2000, 20).

In order to directly apply the new interpretation to Marx’s reproduction schema, a convenient starting point is provided by the expression for net output shown in equation (3). Turning this expression into an aggregate equation:

$$i'Y = i'hvY + i'F \qquad (8)$$

and hence:

$$y = wlX + f \qquad (9)$$

since $i'h$ represents the wage rate, the total amount of money consumption per unit of labour (assuming zero savings by workers). Now we can write:

$$wlX = w\frac{lX}{y}y = w\lambda_m y \qquad (10)$$

Substituting (10) into (9) it follows that

$$y = \lambda_m wy + f \qquad (11)$$

or

$$y = \frac{1}{1-\lambda_m w} f = \frac{1}{e^*} f \qquad (12)$$

Using the new interpretation an aggregate multiplier can be derived from multisectoral foundations that has a clear role for the surplus value term e^* in the denominator. The denominator of this multiplier is once again the per capita share of surplus value, but defined according to the new interpretation of the value of labour power. The VLP expression $\lambda_m w$ is also the propensity to consume derived from multisectoral foundations. This aggregate multiplier is derived without making any restrictive assumptions about the proportionality between prices and values. In the new interpretation the proportionality between money and value spheres is established at an aggregate level by the specification of the value of money.

The Domar Growth Model

The multiplier required to set up Domar's Growth Model is defined with respect to investment and aggregate income. Following Kalecki (1971) this relationship can be established by assuming that aggregate gross profits (P) are determined by total final demand (f), which is made up of aggregate investment (I) and capitalists' consumption (u):

$$P = u + I \qquad (13)$$

Assuming that capitalist consumption is proportional to profits $(u = \lambda P)$, it follows that:

$$P = \frac{1}{1-\lambda} I \qquad (14)$$

And since $P = f$ the multiplier relationship in equation (12) can be re-expressed to show the relationship between income and investment:

$$y = \frac{1}{e^* \lambda^*} I \qquad (15)$$

where $\lambda^* = 1 - \lambda$, the ratio of investment to profits. The expression $e^*\lambda^*$ is the propensity to consume decomposed according to the microfoundations associated with Marx's reproduction schema.

Central to Domar's (1957) model of economic growth is the specification of an aggregate multiplier relationship between changes in income (Δy) and changes in investment (ΔI). For our purposes this can take the form:

$$\Delta y = \frac{\Delta I}{e^* \lambda^*} \qquad (16)$$

Alongside this modelling of demand side relationships, Domar captures the supply side by defining σ as the productivity of investment, the economy's capacity to increase income in proportion to the increase in capital stock. It follows that

$$\sigma = \frac{\Delta y}{I} \qquad (17)$$

since investment represents the change in capital stock.

Domar (1957, p. 87) assumes, along with Marx in *Capital* Volume 2, that there is full capacity utilisation. Bringing together equations (16) and (17), with the restrictive assumption that under full employment of labour the potential change of output matches the change in output demand by the economy:

$$\frac{\Delta I}{\lambda^* e^*} = I\sigma \qquad (18)$$

If both sides of (18) are divided by I and multiplied throughout by $\lambda^* e^*$ it follows that:

$$\frac{\Delta I}{I} = \lambda^* e^* \sigma \qquad (19)$$

With income a constant multiple of investment (see equation 15) it follows that the rate of change of investment is equal to the rate of change of income:

$$\frac{\Delta Y}{Y} = \frac{\Delta I}{I} = \lambda^* e^* \sigma \qquad (20)$$

The full employment rate of growth would, in the unlikely event that this could be achieved, be equal to the multiple of λ^* (the ratio of investment to profits), e^* (the per capita share of surplus value) and σ (the productivity of investment).

This model can be applied to Marx's reproduction schema by specifying each of the three core parameters. Table 3 shows the expanded schema of Table 1 in a form that enables these parameters to be specified. First, $e^* = \lambda_m w = 1 - 1/2 = 1/2$, since the wage rate is equal to $1/2$ and the value money is equal to 1.[4] Second the ratio of investment to profits can be calculated, for example in Year 4, as $\lambda^* = \frac{I}{P} = \frac{955}{2299} = 0.415$. And third, the productivity of investment in Year 4 takes the value $\sigma = \frac{\Delta y}{I} = \frac{418}{869} = 0.481$. The balanced growth rate in Table 3 therefore takes the value:

$$\lambda^* e^* \sigma = 0.415 \times 0.5 \times 0.481 = 0.1 \qquad (21)$$

Since prices and values are equivalent in Tables 1 to 3 this balanced growth result could be established using either of the multiplier relationships in equations (6) or (12), with the value of labour power defined in terms of embodied labour or according to the ‘new interpretation’. The advantage of the new interpretation is that this balanced growth result can be established with prices deviating from values.

4. Conclusions

This paper provides a derivation of the well-known Domar condition for balanced economic growth. Using Marx’s reproduction schema as a starting point the first step in this analysis is to establish the role of a macro multiplier relationship in the schema. This derivation is achieved by transforming the two-department schema to three departments, following Kalecki, and interpreting the tables from a Leontief input-output perspective.

Since the generality of Marx’s schema is limited by the restrictive assumption that prices and values are proportional, a more flexible multiplier relationship is required. The key contribution of the paper is to re-cast the aggregate multiplier relationship according to the ‘new interpretation’ of Marxian economics, developed by Foley (1982) and Lipietz (1982). Using this aggregate multiplier relationship a translation is provided between the reproduction schema and the Domar growth model. In contrast to usual one-sector versions of the Domar model, a macroeconomic framework for modelling economic growth is developed that is consistent with multisectoral foundations.

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Table 1 Marx's Expanded Reproduction Schema

Year 1	C_i	V_i	S_i	X_i
Dept. I[5]	4000	1000	1000	6000
Dept. II	1500	750	750	3000
	5500	1750	1750	9000

Year 2	C_i	V_i	S_i	X_i
Dept. I	4400	1100	1100	6600
Dept. II	1600	800	800	3200
	6000	1900	1900	9800

Year 3	C_i	V_i	S_i	X_i
Dept. I	4840	1210	1210	7260
Dept. II	1760	880	880	3520
	6600	2090	2090	10780

Year 4	C_i	V_i	S_i	X_i
Dept. I	5324	1331	1331	7986
Dept. II	1936	968	968	3872
	7260	2299	2299	11858

Year 5	C_i	V_i	S_i	X_i
Dept. I	5856	1464	1464	8784
Dept. II	2129	1065	1065	4259
	7985	2529	2529	13043

Table 2 Marx's Reproduction Scheme as an Input-Output Table

(a) Numerical Representation

Year 1	Dept.1	Dept. 2	Dept. 3	dC	dV	u	X_i
Dept. 1	4000	550	950	500			6000
Dept. 2						1100	1100
Dept. 3	1000	275	475		150		1900
S_i	1000	275	475				
X_i	6000	1100	1900				9000

(b) Algebraic Representation

Year 1	Dept.1	Dept. 2	Dept. 3				
Dept. 1	$a_{11}X_1$	$a_{12}X_2$	$a_{13}X_3$	dC			X_1
Dept. 2						u	X_2
Dept. 3	$h_3l_1X_1$	$h_3l_2X_2$	$h_3l_3X_3$		dV		X_3
	S_1	S_2	S_3				
	X_1	X_2	X_3				

Table 3 Rates of Growth in the Expanded Reproduction Schema

Periods	Constant Capital	Variable Capital	Profits	Net Income	$\frac{\Delta Y}{Y}$	I	$\frac{\Delta I}{I}$
1	5500	1750	1750	3500	–	–	–
2	6000	1900	1900	3800	0.09	650	–
3	6600	2090	2090	4180	0.1	790	0.22
4	7260	2299	2299	4598	0.1	869	0.1
5	7985	2529	2529	5058	0.1	955	0.1

Units are in £ sterling

Footnotes

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[1] The algebraic components of the model have the following numerical values in Table 2a:

$$a_{11} = \frac{4000}{6000} = \frac{2}{3}, \quad a_{12} = \frac{550}{1100} = \frac{1}{2}, \quad a_{13} = \frac{950}{1900} = \frac{1}{2},$$

$$l_1 = \frac{2000}{6000} = \frac{1}{3}, \quad l_2 = \frac{550}{1100} = \frac{1}{2}, \quad l_3 = \frac{950}{1900} = \frac{1}{2},$$

$L = 2000 + 550 + 950 = 3500$, and

$$h_3 = \frac{1750}{3500} = \frac{1}{2}.$$

For example, a flow of $a_{13}X_3$ capital goods from Department 1 to 2 is calculated as $\frac{1}{2} \times 1900 = 950$. Similarly,, the flow of wage goods $h_3 l_2 X_2$ consumed by workers in department 2 is calculated as $\frac{1}{2} \times \frac{1}{2} \times 1100 = 275$.

[2] The amount of direct labour power employed is equal to the total net income of the economy: $L = vY = i'Y = y$. To prove that $v = i'$ in Table 2a:

$$v = l(I - A)^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 & \frac{3}{2} & \frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

[3] Following the notation used in Section 2 the term y continues to be a scalar representing money net output. In expositions of the new interpretation this is usually written explicitly as pq, the product of a price vector (p) and a column vector of physical net outputs (q).

[4] In Table 2 for each unit of labour performed, half of the effort is remunerated in the form of variable capital. Hence the wage rate is $1/2$. It can also be seen in Table 2 that for Year 1 the value of money takes the value $\lambda_m = \frac{lX}{y} = \frac{3500}{3500} = 1$.