Endogenous growth in a multi-sector economy

by

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1. Introduction

There are at least three different approaches to endogenous growth (Jones and Manuelli, 1997, survey this literature). Two include nonconvexities or externalities or both. The third relies on convex models of growth in which, properly interpreted, the two welfare theorems hold (e.g., Jones and Manuelli, 1990 and Rebelo, 1991). The models in this last strand of literature are characterized by the fact that production is not limited by primary resources, so that even if the technology is stationary, unbounded production paths are feasible. For the sake of simplicity we call "convex models" the models whose equilibria satisfy the conditions of the two Fundamental Theorems of Welfare Economics, and subdivide them in "bounded models" (those whose feasible paths are limited by the availability of natural resources) and "unbounded models". In the last fifteen years, models with explicit consumption and a production side in which 'goods are made out of goods alone' (Champernowne, 1945, p. 12) has been widely used in the new growth theory, especially in that approach to endogenous growth based on the assumption that all production factors are reproducible (Lucas, 1988; Rebelo, 1991). It could be argued that any mechanism the literature has envisaged to make sustained growth possible has essentially involved the assumption that there is a 'core' of capital goods whose production does not require (either directly or indirectly) non-producibile factors. In fact the reduced form of most endogenous growth models is linear or asymptotically linear in the reproducible factors (see for example, Frankel, 1962; Romer, 1986, 1987, 1990)[1]. A consequence has been that in the

[1] Some theorists even came to the conclusion that unbounded growth is more an assumption about the linearity of the technology than a result of the models; see for instance Romer, 1990, p. S84.

endogenous growth literature the static analysis is mainly centered around the concept of 'balanced growth path', which, in this context, plays the role played by the stationary state in convex bounded models.

It is common in the literature on growth to study low-dimensional models (one- or two-sector models) with special features. For example, Rebelo (1991) studies two different one-capital good models: the textbook AK model and a two sector model in which the capital good is produced without nonreproducible factors, but a nonreproducible factor in fixed supply is used in the production of the consumption good. Lucas (1988), Romer (1990) and many others, on the other hand, study two-capital goods models. The exceptions are some convex models. Recent contributions to *n*-sector unbounded growth models are contained in Dasgupta and Mitra (1988), Dolmas (1996), Alvarez and Stokey (1998), Kaganovich (1998), Ossella (1999) and Jensen (2000), who work in a discrete time setting, and in Freni, Gozzi and Salvadori (2001, hereafter FGS), who consider instead a continuous time framework.

In FGS we studied a multisector 'AK model' in continuous time. We provided an existence result for optimal strategies, a set of duality results, and a complete classification of the price supported steady states of the model. In doing so we used a number of assumptions. In the present paper we will argue about removing some of those assumptions. Moreover, since this paper is designed for a less mathematical oriented audience, we provide several interpretative elements, relevant in dealing with convex unbounded models.

The various building blocks of the model presented in FGS have different origins. In particular, the production side of the model has a clear classical flavor (see Kurz and Salvadori, 1995) and is very close to the production side of the von Neumann model, in which commodities are produced out of each other because '[w]age costs are not considered as such, for laborers are not separately considered any more than are farm animals' (Champernowne, 1945, p. 12). This structure hints to a 'technological' theory of the long run rate of interest, that 'appears as the natural and optimum rate of organic expansion of the system, and depends on the technical processes of production which are available' (*ibid.*).

Nevertheless, the model has also a Ramsey-like preference side in which the optimal behavior of the representative agent determines the system's saving rate. Since we maintain the hypothesis that the behavior of the representative agent does not affect the technical conditions of production, our rational agent cannot be a worker, because the real wages is still 'whatever is needed to persuade people to work' (*ibid*., p. 16). Therefore, it is '[t]he question of consumption by the propertied class' (*ibid*.) that properly arises in the model with an intertemporal utility functional. However, even under this 'representative capitalist' interpretation, the introduction of explicit consumption creates a tension in the model as regards the forces determining the long run rate of interest.

The Classification Theorem in FGS, which we report here in section 3, resolves the question under the hypotheses of (i) single production, (ii) the existence of a unique indecomposable technique, and (iii) a single consumption good. It states that if the discount rate is low enough to induce positive production of all commodities, then the rate of profit and the relative prices are those prevailing in the von Neumann equilibrium. At the other extreme, if the discount rate is high enough, the pure capital goods are free goods and the system collapses to the one-sector model in which the consumption good is produced by itself and free goods. For intermediate values of the discount rate, on the contrary, changes in preferences affect both the rate of profit and the relative prices. Ultimately, therefore, both considerations of technology-and-cost arbitrage and preferences concur in determining the growth rate, the profit rate and the relative prices that prevail in the long run equilibrium. Nevertheless, in the model there is some space for the more 'classical' opinion that 'even if part of the income from property were spent on consumption, and not saved, the rate of interest would not necessarily be much affected: it might still be *approximately* equal to the greatest expansion rate that *would* have been possible *if* all income from property had been saved' (*ibid*.).

Next two sections are preliminary. In Section 2 we present the model in a general format whereas in Section 3 we present the results obtained by FGS in a restrictive setting. Section 4 is devoted to a comparison with the von Neumann-Sraffa-Morishima models developed mainly in the sixties and the seventies. This section clarifies also what are the extensions of the results by FGS which, on the basis of the comparison, are expected to be easily obtained and what should require more effort. Sections 5-7 are mainly devoted to clarify the differences in the analysis connected with relaxing some assumptions: decomposability of the input matrix (Section 5), joint production (Section 6), multiple consumption goods (Section 7). Section 8 provides some conclusions.

2. A class of convex multisector endogenous growth models in continuous time

In this section we describe the multisector model which we aim to analyze and which we analyzed under special assumptions in previous papers (Salvadori, 1998; FGS; Gozzi and Freni, 2001). In next section we summarize the main static and comparative statics results presented in FGS. Although the production framework used in FGS is the 'simple linear production model' (Gale, 1960, p. 294) which excludes both joint production and choice of technique, the complete set of results we summarize in the following Theorem 3 (the Classification Theorem) is not known in the literature. The reason is that the study of multisector closed linear models of production with explicit consumption has been confined almost exclusively to the discrete-time framework (see McFadden, 1967, 1973; Atsumi, 1969 and the more recent literature we mentioned in section 1), while we study a continuous-time model. As a consequence some kinds of complications which in discrete time can be avoided at first (see however Atsumi, 1969, p. 270), arise from the beginning. These complications are generally connected with joint production and decomposability, but in a continuous-time model they can also be connected with the way fixed capital is formalized in order to avoid an infinite number of commodities (depreciation by evaporation).

Consider a 'representative agent' economy described by an aggregate technology, an instantaneous utility function, u, and a discount rate, $\rho \in \Re$. The commodity space is finite and the technology is stationary. In particular, we assume that no pure consumption good is present in the system, while there is a number n, $n \geq 1$, of capital stocks, each of which continuously generates a flow of productive services proportional to the amount of the stock. Production consists in combining the productive services from the stocks to generate flows that add to the existing stocks. Decay and consumption, on the other hand, drain away the stocks. For the sake of simplicity, we assume that the rate of decay of the stocks in production is given by a single constant δ_x, $\delta_x \geq 0$. Stocks not used in production are 'stored'. Stored capitals decay at the rate δ_z, $\delta_z \geq 0$. To avoid jumps in the stocks we conceptualize 'disposal' as a storage process with $\delta_z \geq \delta_x$. Each stock can be produced and no productive service from primary factors is relevant in the system.

The production set is generated by a finite number m, $m \geq 1$, of independent activities (or processes), each of which can be run at any scale of operation. Hence, process j (j = 1, 2, ..., m) can be represented by a pair of n- dimensional input-output vectors:

$$\mathbf{a}_j^T \to \mathbf{b}_j^T,$$

where $\mathbf{a}_j^T\mathbf{e}_i \geq 0$ is the amount of productive services from stock *i* that process *j* uses at the unitary level of activation and $\mathbf{b}_j^T\mathbf{e}_i \geq 0$ is the flow of commodity *i* produced by the same process at the same level of activation. Thus, we can summarize the production processes with a pair of *m*x*n* non-negative matrices in the following way:

$$\mathbf{A} \rightarrow \mathbf{B}.$$

The way we handle storage (or disposal) processes implies that we implicitly assume that, in addition to the *m* processes formally included in the technology, free disposal activities for the productive services of the stocks are available.

We give a basic characterization of the technology by means of the following two classical von Neumann-like assumptions (Gale, 1960, p. 311):

[HP1] Each column of matrix **B** is semipositive.

This assumption means that all commodities are reproducible and, therefore, there is no primary factor. If this assumption does not hold and an existing commodity *j* cannot be produced, then commodity *j* can be interpreted as a kind of an exhaustible resource.

[HP2] Each row of matrix **A** is semipositive.

This assumption means that no process can be activated without using (the service of) some commodity as an input. It, therefore, implies that for each $t \geq 0$ the intensity levels of the production processes are bounded from above by the existing stocks. It is convenient to assume that the system can grow at a positive rate since this case is the most interesting one from an economic point of view:

[HP3] $\exists\, \mathbf{x} \geq \mathbf{0}, g > 0 : \mathbf{x}^T\left[\mathbf{B} - \left(\delta_x + g\right)\mathbf{A}\right] \geq \mathbf{d}^T$, where $\mathbf{d}$ is any nonnegative vector proportional to the vector of consumed commodities.

FGS and Gozzi and Freni (2001) concentrate on the case in which there is no joint production in flow outputs. That is:

[HP4] Each row of matrix **B** has one and only one positive element.

Note that in this case each positive element of **B** can normalized to 1 without loss of generality. Under Assumption [HP4] processes can be unambiguously linked to industries. If, moreover, each industry has one production process, then the input/output matrices are square and we will have:

[HP4.1] $m = n$ (i. e. **A** is square and **B** = **I**).

Another assumption about production we sometime use is:

[HP5] $\mathbf{x}^T\left[\mathbf{B}-\left(\delta_x+g\right)\mathbf{A}\right]\geq\mathbf{0}^T, g>-\delta_x, \mathbf{x}\geq\mathbf{0}, \mathbf{x}\neq\mathbf{0}\Rightarrow\mathbf{x}^T\mathbf{B}>\mathbf{0}^T$

It means that the services from each stock enter directly or indirectly into the production of each commodity (cf. Kurz and Salvadori, 1998, pp. 95-7). In particular, if Assumptions [HP4] and [HP4.1] holds, [HP5] is equivalent to the assumption:

[HP4.2] The square matrix **A** is indecomposable.

As mentioned above, the way disposal is conceptualized implies the assumption

[HP6] $\delta_z \geq \delta_x$.

The discount rate, $\rho\in\Re$, and the instantaneous utility function, u, describe the preference side of the economy because, as it is quite common in the new growth theory (exceptions are Ben-Gad, 1999, and Jensen, 2000, who consider heterogeneous agents), we deal with a single-consumer economy whose preferences can be represented by a utility functional $U(\mathbf{c}(t))$ having the form

$$U(\mathbf{c}(t))=\int_0^\infty e^{-\rho t}u(\mathbf{c}(t))dt,$$

where

$$u(\cdot):\Re_+^{c_m}\rightarrow\Re\cup\{-\infty\}, 1\leq c_m\leq n\ .$$

Note that if $c_m<n$, then there are $n-c_m$ pure capital goods. Since the production side of the model is linear, the whole model is homogeneous if the preference side is so. We therefore assume that the utility function is the usual iso-elastic one:

[HC1] $u(\cdot)=\dfrac{1}{1-\sigma}[v(\cdot)]^{1-\sigma}\quad for\ \sigma>0,\ \sigma\neq1,\ u(\cdot)=\log[v(\cdot)]\quad for\ \sigma=1;$

[HC2] $v(\cdot): \Re_+^{c_m} \to \Re_+$ is increasing, concave and homogeneous of degree one. If $c_m > 1$, then $v(\cdot)$ is strictly concave.[2]

Preferences are fully described by ρ, σ, and function $v(\cdot)$. In the following we will say that $v(\cdot)$ describes the preferences concerning consumption at a given moment in time, whereas ρ and σ describe preferences concerning distribution of consumption (and saving) over time. Note that if $c_m = 1$, the utility function $v(\cdot) := c$ is equivalent to any other. If $c_m > 1$, examples of $v(\cdot)$ are the Cobb-Douglas function, $v(\mathbf{c}) = \prod_i c_i^{\theta_i}, \sum_i \theta_i = 1$, and the CES function, $v(\mathbf{c}) = \left(\sum_i \alpha_i c_i^{\theta} \right)^{1/\theta}$, $\theta < 1$. Hence, if

[HC3] $c_m = 1$,

then the two parameters ρ and σ completely describe the preference side of the model. FGS analyzed a case in which $c_m = 1$. More general cases are considered in Gozzi and Freni (2001).

Let $\mathbf{s}$ be the *n*x1 vector of stocks, **x** be the *m*x1 intensity vector and $\hat{\mathbf{c}}$ be the *n*x1 vector obtained from the consumption vector $\mathbf{c}$ by adding a zero component for each pure capital good. The evolution of the stocks is given by the following differential equation:

$$\dot{\mathbf{s}}^T = \mathbf{x}^T\mathbf{B} - \delta_x \mathbf{x}^T\mathbf{A} - \delta_z(\mathbf{s}^T - \mathbf{x}^T\mathbf{A}) - \hat{\mathbf{c}}^T$$

with the constraints

$$\mathbf{x}^T\mathbf{A} \leq \mathbf{s}^T, \mathbf{x} \geq \mathbf{0}, \mathbf{c} \geq \mathbf{0}.$$

The common approach to competitive equilibria analysis in the above setting is through the extension of the first and second welfare theorems for finite dimensional economies. This strategy leads to investigate the link between the competitive equilibria of the system and the solutions, if there are any, to the problem:

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2 The use of iso-elastic utility functions goes back to Ramsey (1928) who studied this 'interesting special case' in section II of his 1928 paper.

$$
\begin{aligned}
V(\mathbf{s}^*) &= \sup \int_0^\infty e^{-\rho t} u(\mathbf{c}(t))dt \\
\dot{\mathbf{s}}^T &= \mathbf{x}^T\mathbf{B} - \delta_x \mathbf{x}^T\mathbf{A} - \delta_z(\mathbf{s}^T - \mathbf{x}^T\mathbf{A}) - \hat{\mathbf{c}}^T \\
&\mathbf{x}^T\mathbf{A} \le \mathbf{s}^T, \mathbf{x} \ge \mathbf{0}, \mathbf{c} \ge \mathbf{0} \\
&\mathbf{s}(0) = \mathbf{s}^* \ge \mathbf{0} \quad \textit{given}.
\end{aligned} \tag{P}
$$

We will therefore be interested in the existence and characterization of the paths for which the sup in (P) is attained. Moreover, since under the above assumptions (P) is an homogeneous program, our interest will lie also in the existence and characterization of the special paths that solve (P) and enjoy a steady state structure. Of course, these paths provide the simplest reference point for the analysis of the asymptotic behavior of non stationary paths.

In studying the optimal control problem (P), the Hamiltonian formalism is often used to introduce the 'price' variables. From an economic point of view, this is a particularly significant procedure because it leads to the introduction of competitive prices. Indeed, what are defined as competitive paths are simply stock-price paths supporting the maximized Hamiltonian (see Cass and Shell, 1976). In our context, since the optimal control problem (P) is autonomous, the discounted Hamiltonian is used. It is given by:

$$
H^D(\mathbf{s},\mathbf{v}) = \max_{\mathbf{c}\ge\mathbf{0}}\left[u(\mathbf{c}) - \hat{\mathbf{c}}^T\mathbf{v}\right] - \delta_z \mathbf{s}^T\mathbf{v} + \max_{\substack{\mathbf{x}^T\mathbf{A}\le\mathbf{s}^T \\ \mathbf{x}\ge\mathbf{0}}} \mathbf{x}^T[\mathbf{B} - (\delta_x - \delta_z)\mathbf{A}]\mathbf{v}.
$$

It is just the case to remark that the linear programming problem involved in the definition of $H^D(\mathbf{s},\mathbf{v})$ requires the existence of a vector $\mathbf{q}^*(\mathbf{s},\mathbf{v}) \in \Re^n_+$ which is a solution to the dual problem and can be interpreted as the vector of the equilibrium "rentals" rates for the use of the stocks. Hence, if $\mathbf{c}^*(\mathbf{v}), \mathbf{x}^*(\mathbf{s},\mathbf{v})$ indicate a set of controls solving the max problems involved in the definition of $H^D(\mathbf{s},\mathbf{v})$, then the set of paths $[\mathbf{s}(t),\mathbf{v}(t),\mathbf{c}^*(\mathbf{v}(t)),\mathbf{x}^*(\mathbf{s}(t),\mathbf{v}(t)),\mathbf{q}^*(\mathbf{s}(t),\mathbf{v}(t))]$, which satisfy appropriate continuity properties and solve

$$
\dot{\mathbf{s}} \in \partial H^C_{\mathbf{v}}(\cdot,\cdot) \tag{1}
$$

$$
\dot{\mathbf{v}} - \rho\mathbf{v} \in -\partial H^C_{\mathbf{s}}(\cdot,\cdot) \tag{2}
$$

is called a *competitive program*, i.e. a critical point for the problem (P). A transversality condition is then involved in the extension of the first and second welfare theorems to infinite horizon economies. The first welfare theorem, in particular, will state that absolutely continuous competitive paths with non negative prices that satisfy a suitable transversality condition are optimal. It is well known, however, that a full converse of this result does not hold due to the possibility that absolutely continuity of prices cannot be granted for stock paths hitting the non negativity boundary. Moreover, prices supporting non interior stock paths can fail to exist altogether (examples of this phenomenon are given in FGS, Appendix D).

In closing this section we point out that there are many convex endogenous growth models that cannot be reduced at once to the present framework. We mention a few of them. First, models with pure consumption goods and/or nonreproducible resources that are not essential (see e.g. Bose, 1968; Weitzman, 1971; Jones and Manuelli, 1990; Rebelo, 1991). Second, models with adjustment costs (see e.g. Burgstaller, 1994; Dolmas, 1996; Ladron-de-Guevara, Ortigueira and Santos, 1999). Third, models in which the technology is not polyhedral (e.g. Kaganovich, 1998, Jensen, 2000). Finally, models with an infinite dimensional commodity space (Boldrin and Levine, 2000, 2001; Boucekkine, Licandro, Puch and del Rio, 2000).

3. Steady states in a 'simple' multisector AK model

In FGS Assumptions [HP1], [HP2], [HP3], [HP4], [HP4.1], [HP4.2], [HP6], [HC1], [HC2], [HC3] hold. Moreover, it is also assumed that all commodities are available at time 0 ($\mathbf{s}^* > \mathbf{0}$) and that the (unique) consumption good enters directly in its own production. If the consumption good is commodity 1, this means

[HP7] $a_{11} > 0$.

The first of these two assumptions implies that there is an admissible solution to problem (P) with a positive $\mathbf{s}$ for each $t > 0$. The second assumption guarantees that $\mathbf{s} > \mathbf{0}$ for each $t > 0$ in each optimal solution starting at any $\mathbf{s}^* > \mathbf{0}$. This result is relevant since if $\mathbf{s} > \mathbf{0}$, then $\mathbf{q}$ is bounded and, therefore, prices $\mathbf{v}$ cannot jump.

FGS first prove that an optimal path exists if and only if

[HE] $\sigma\Gamma > \Gamma - \rho$

where

$$\Gamma = \sup \{g : \exists\, \mathbf{x} \geq \mathbf{0}: \mathbf{x}^T\left[\mathbf{B} - \left(\delta_x + g\right)\mathbf{A}\right] \geq \mathbf{e}_1^T\}$$

and $\mathbf{e}_1$ is the first unit-vector: a vector proportional to the vector of consumed commodities.[3] Under the assumptions maintained by FGS $\Gamma = \lambda_{PF}^{-1} - \delta_x$, where λ_{PF} is the Perron-Frobenius eigen-value of matrix **A**.

The above existence theorem is completed with two theorems concerning the optimality conditions for the problem at hand, that are the extensions to the present framework of standard results holding in smooth-bounded models. In particular it is proved that a competitive program is optimal if the usual condition that the value of the stocks converges to zero holds. Formally we have the following:

Theorem 1. Let Assumptions [HP1], [HP2], [HP3], [HP4], [HP4.1], [HP4.2], [HP6], [HP7] [HC1], [HC2], [HC3] hold. Let $[\mathbf{s}(t), \mathbf{v}(t), \mathbf{c}^*(\mathbf{v}(t)), \mathbf{x}^*(\mathbf{s}(t), \mathbf{v}(t)), \mathbf{q}^*(\mathbf{s}(t), \mathbf{v}(t))]$ be a competitive program starting at $\mathbf{s}^* > \mathbf{0}$ such that $[\mathbf{s}(t)^T, \mathbf{v}(t)^T] \geq \mathbf{0}^T$ is absolutely continuous, $\mathbf{c}^*(\mathbf{v}(t))$, $\mathbf{x}^*(\mathbf{s}(t), \mathbf{v}(t))$, $\mathbf{q}^*(\mathbf{s}(t), \mathbf{v}(t))$ are measurable and locally integrable, and

$$\lim_{t \to \infty} e^{-\rho t} \mathbf{s}(t)^T \mathbf{v}(t) = 0.$$

Then $[\mathbf{s}(t), \mathbf{c}^*(\mathbf{v}(t)), \mathbf{x}^*(\mathbf{s}(t), \mathbf{v}(t))]$ is an optimal solution of problem (P).

Although the same result can be established under much weaker assumptions, we stated Theorem 1 in this form in view of the converse.

Theorem 2. Let Assumptions [HP1], [HP2], [HP3], [HP4], [HP4.1], [HP4.2], [HP6], [HP7] [HC1], [HC2], [HC3] hold. If $[\hat{\mathbf{s}}(t), \hat{\mathbf{c}}(t), \hat{\mathbf{x}}(t)]$ is an optimal solution of problem (P) starting at $\mathbf{s}^* > \mathbf{0}$, then there is an absolutely continuous path $\mathbf{v}(t) \geq \mathbf{0}$ and a measurable and locally integrable path $\mathbf{q}^*(\mathbf{s}(t), \mathbf{v}(t)) \geq \mathbf{0}$ such that $[\mathbf{s}(t), \mathbf{v}(t), \mathbf{c}^*(\mathbf{v}(t)), \mathbf{x}^*(\mathbf{s}(t), \mathbf{v}(t)), \mathbf{q}^*(\mathbf{s}(t), \mathbf{v}(t))]$ is a competitive program, $\mathbf{c}^*(\mathbf{v}(t)) = \hat{\mathbf{c}}(t)$, $\mathbf{x}^*(\mathbf{s}(t), \mathbf{v}(t)) = \hat{\mathbf{x}}(t)$, and

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3 We note that the utility function u is unbounded above or below or, in the log case, both above and below. Moreover, the boundedness of feasible paths is not assumed. This generates a non-trivial existence problem for (P) which is usually solved by the introduction of an existence condition linking the technology with the preferences (see for example McFadden, 1967, 1973; Brock and Gale, 1969, and Atsumi, 1969).

$$\lim_{t\to\infty} e^{-\rho t}\mathbf{s}(t)^T \mathbf{v}(t) = 0 .$$

As a step towards the study of the dynamics of the system FGS analyzed the *steady-state optimal solutions* to problem (P), which are defined as optimal solutions $[\mathbf{s}(t), \hat{\mathbf{c}}(t)), \hat{\mathbf{x}}(t)]$ to problem (P) for which there is a real number *g*, a real number c_0, and a nonnegative vector $\mathbf{x}_0$ such that $\hat{\mathbf{c}}(t) = c_0 e^{gt}\mathbf{e}_1$, $\hat{\mathbf{x}}(t) = \mathbf{x}_0 e^{gt}$, where $\mathbf{e}_1$ is the first unit vector. Let us remark that the definition of steady-state optimal solutions does not state that relative prices are constant over time. An example can clarify the issue. Let (P) be the problem:

$$V(\mathbf{s}^*) = \sup \int_0^\infty 2e^{-\frac{1}{3}t} c^{\frac{1}{2}} dt$$
$$\dot{\mathbf{s}}^T = \mathbf{x}^T - \tfrac{1}{2}\mathbf{x}^T\mathbf{A} - \tfrac{1}{2}(\mathbf{s}^T - \mathbf{x}^T\mathbf{A}) - c\mathbf{e}_1^T$$
$$\mathbf{x}^T\mathbf{A} \le \mathbf{s}^T, \mathbf{x} \ge \mathbf{0}, \mathbf{c} \ge \mathbf{0}$$
$$\mathbf{s}(0) = \mathbf{s}^* .$$

where $\mathbf{A} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{bmatrix}$ and $\mathbf{s}^* = \begin{bmatrix} \frac{42}{31} \\ \frac{72}{31} \\ \frac{72}{31} \end{bmatrix}$. It is easily checked that

$$\mathbf{s} = \mathbf{s}^* e^{\frac{1}{3}t},\ \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{-\frac{1}{6}t} + h\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{-\frac{7}{6}t},\ c = e^{\frac{1}{3}t},\ \mathbf{x} = \begin{bmatrix} \frac{66}{31} \\ \frac{60}{31} \\ \frac{60}{31} \end{bmatrix} e^{\frac{1}{3}t},\ \mathbf{q} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{-\frac{1}{6}t} + 2h\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{-\frac{7}{6}t},$$

for -1/2 < h < 1/2 is a *steady-state optimal solution* to problem (P). In this example, despite the fact that the quantity side grows at rate 1/3, the relative prices and relative rentals do not need to be constant (they are so if and only if h = 0). However, FGS have proved that for each steady state solution there are a price path and a rental path such that

$$\mathbf{v} = \mathbf{v}(0)e^{-g\sigma t},\ \ \mathbf{q} = (\rho + \delta_z + g\sigma)\mathbf{v}(0)e^{-g\sigma t}.$$

We will refer to these prices and rentals as *steady state price-rental paths*. On the basis of these and other results, FGS reduce the problem of the optimal steady states to the analysis of finding scalars $g \in [\frac{1}{\sigma}(\lambda_{PF}^{-1} - \delta_x - \rho), \lambda_{PF}^{-1} - \delta_x)$ and c_0 and vectors $\mathbf{s}_0$, $\mathbf{x}_0$, and $\mathbf{v}_0$ such that:

$$\mathbf{e}_1^T\mathbf{v}_0 = c_0^{-\sigma} \tag{3.1}$$

$$[\mathbf{I} - (\rho + \delta_x + g\sigma)\mathbf{A}]\mathbf{v}_0 \le \mathbf{0} \tag{3.2}$$

$$\mathbf{x}_0^T[\mathbf{I}-(\rho+\delta_x+g\sigma)\mathbf{A}]\mathbf{v}_0=\mathbf{0} \tag{3.3}$$

$$\mathbf{x}_0^T\mathbf{A}-\mathbf{s}_0^T\leq\mathbf{0} \tag{3.4}$$

$$[\mathbf{x}_0^T\mathbf{A}-\mathbf{s}_0^T]\mathbf{v}_0=\mathbf{0} \tag{3.5}$$

$$\mathbf{x}_0^T[\mathbf{I}-(g+\delta_x)\mathbf{A}]\mathbf{v}_0=c_0\mathbf{e}_1^T\mathbf{v}_0 \tag{3.6}$$

$$\mathbf{x}_0^T[\mathbf{I}-(g+\delta_x)\mathbf{A}]\geq c_0\mathbf{e}_1^T \qquad \text{for } g+\delta_z\geq 0 \tag{3.7}$$

$$\mathbf{s}_0^T-\frac{1}{g+\delta_z}\{\mathbf{x}_0^T[\mathbf{I}-(\delta_x-\delta_z)\mathbf{A}]-c_0\mathbf{e}_1^T\}\geq\mathbf{0} \qquad \text{for } g+\delta_z<0 \tag{3.8}$$

$$c_0>\mathbf{0},\ \mathbf{v}_0\geq\mathbf{0},\ \mathbf{x}_0\geq\mathbf{0}. \tag{3.9}$$

Optimal steady state solutions are obviously relevant when it is possible to prove the convergence of the optimal path towards a steady state solution. However FGS do not report stability results, since preliminary examination of them showed that delicate points arise in the general case. Asymptotic turnpike theorems are known to hold for various discrete-time versions of the linear growth model (see Atsumi, 1969, Dasgupta and Mitra, 1988, Kaganovich, 1998, Ossella, 1999, and Jensen, 2000). Nevertheless, it turned out that the technical issues of non-smoothness and the lack of strict concavity that characterize the continuous-time framework preclude the use of known results even for the analysis of local stability. But there is at least another reason to deal with steady states, that is, that in these states some relevant concepts, such as that of "real rate of profit'" or "growth rate", can be defined. The literature on growth often refers to steady states in order to convey some macroeconomic insights. Lucas (1988), for instance, refers freely to steady state concepts as "the rate of growth" or the "real rate of profit" (that in general are meaningless with regard to non-stationary paths) under the explicit assumption of a fast convergence to the steady state.

As a matter of fact there are profitability concepts which can be used with reference to optimal solutions even if they are not steady states. It is always possible to deal with the "own rate of return of commodity *i*" as the rate of profit which can be obtained by an investment measured in commodity *i* getting e revenue measured in commodity *i* (see, for instance, Malinvaud, 1953). FGS show that in any optimal solution

$$r_i(t) = \rho - \frac{\mathbf{e}_j^T \dot{\mathbf{v}}(t)}{\mathbf{e}_j^T \mathbf{v}(t)}$$

where $r_i(t)$ is the own rate of return of commodity i at time t (this formula is clearly a reminiscence of Fisher formula, when the own rate of return is interpreted as a "real" rate of profit, ρ as a "nominal" rate, and the fraction on the RHS as an inflation rate). FGS proved also that in any optimal solution the growth rate of consumption equals the ratio of the difference between the own rate of return of commodity 1 and ρ over σ:

$$\frac{\dot{c}(t)}{c(t)} = \frac{r_1(t) - \rho}{\sigma}. \qquad (4)$$

In a steady state solution supported by steady states prices all the own rates of return are equal each other so we can call this common rate the "real rate of profit r". Similarly, in a steady state solution all the intensities of operation of processes as well as consumption grow at the same rate so we can call this common rate the "growth rate g". Obviously, in a steady state

$$g = \frac{r - \rho}{\sigma}. \qquad (5)$$

From equations and inequalities (3) FGS obtained a Classification Theorem, in which three different regimes are envisaged, depending on the values of the involved parameters. The Classification Theorem states a particular relationship between the growth rate g and the rate of profit r. This r-g relationship is drawn in Figure 1. In the first regime r is constant and equals $\lambda_{PF}^{-1} - \delta_x$, whereas g varies in the range $(-\delta_x, \lambda_{PF}^{-1} - \delta_x)$. In the second regime g is constant and equals $-\delta_x$, whereas r varies in the range $[\lambda_{PF}^{-1} - \delta_x, a_{11}^{-1} - \delta_x]$. In the third regime r is constant again and equals $a_{11}^{-1} - \delta_x$, whereas g varies in the range $(-\infty, -\delta_x)$. This relationship is not to be confused with another r-g relationship, that is, equation (5). The former depends on technology and on preferences concerning consumption at a given moment in time and does not depend on preferences concerning distribution of consumption over time. The latter, on the contrary, depends only on preferences concerning distribution of consumption over time. In a steady state solution r and g are determined by the intersection between these two r-g relationship. The latter relationship is very known, but also the former is mentioned in the literature. For instance, Evans et al. (1998) refer to it as a "technological" relationship whereas they refer to the latter as a "preferences" relationship. However, as shown in Section 7 below, the former relationship

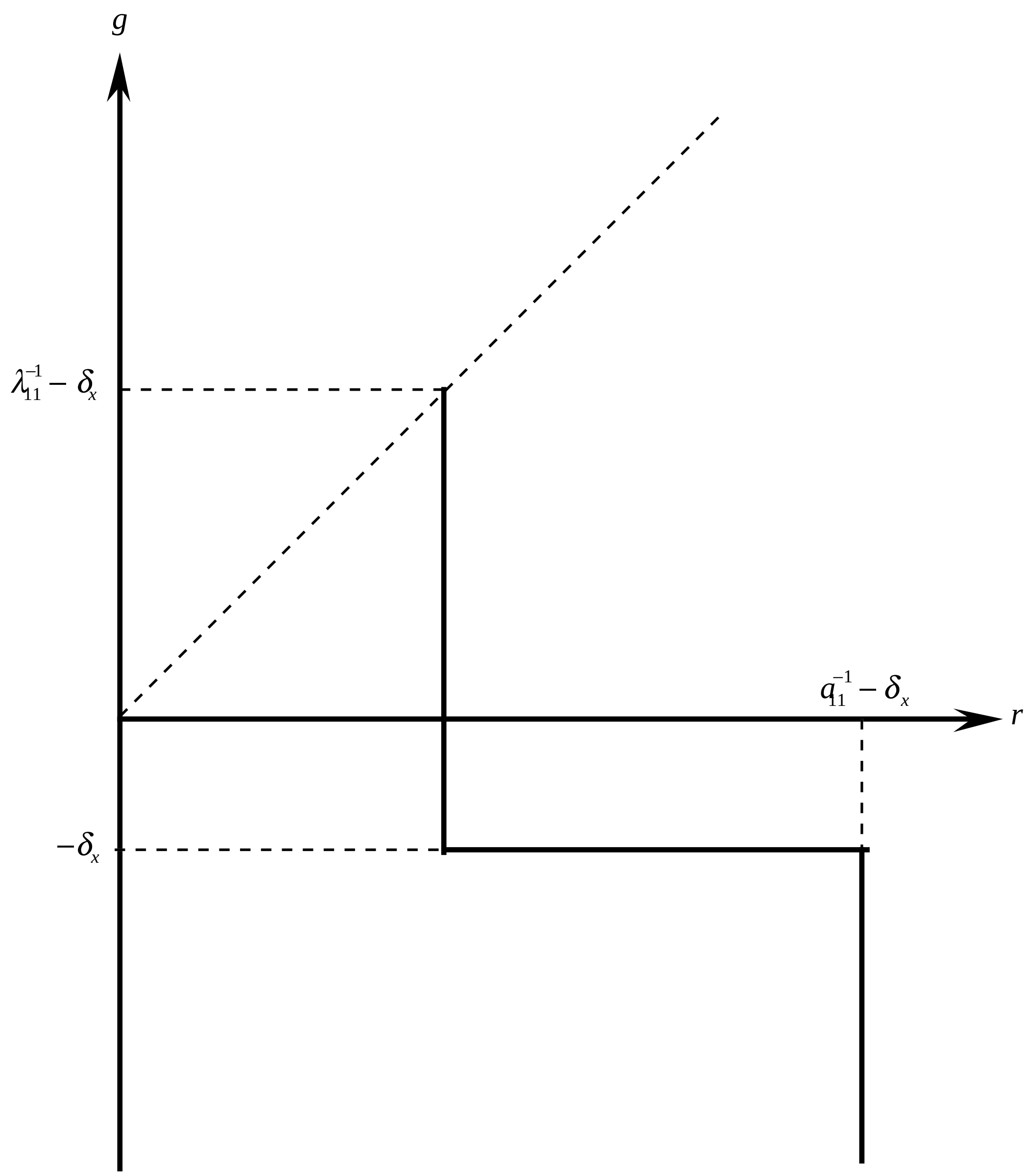


Figure 1

depends not only on technology, but also on preferences concerning consumption at a given moment in time. In formal terms the Classification Theorems reads:

Theorem 3. Let Assumptions [HP1], [HP2], [HP3], [HP4], [HP4.1], [HP4.2], [HP6], [HP7] [HC1], [HC2], [HC3], [HE] hold. Then

(1st regime) If $(\lambda_{PF}^{-1} - \delta_x)(1-\sigma) < \rho < \lambda_{PF}^{-1} - \delta_x(1-\sigma)$, then there is a unique (up to a multiplication by a positive scalar) optimal steady state of Problem (P) given by

$$\mathbf{s}(t) = \mathbf{s}_0 e^{gt},\ c(t) = c_0 e^{gt},\ \mathbf{x}(t) = \mathbf{x}_0 e^{gt}$$

where

$$g = \tfrac{1}{\sigma}(\lambda_{PF}^{-1} - \delta_x - \rho) \in (-\delta_x, \lambda_{PF}^{-1} - \delta_x)$$

$$c_0 > 0,\ \mathbf{x}_0^T = c_0 \mathbf{e}_1^T [\mathbf{I} - (g + \delta_x)\mathbf{A}]^{-1},\ \mathbf{s}_0^T = \mathbf{x}_0^T \mathbf{A} > \mathbf{0}$$

and with supporting prices and rentals given by

$$\mathbf{v} = c_0^{-\sigma} \mathbf{v}_{PF} e^{-g\sigma t},\ \mathbf{q} = (\rho + \delta_z + g\sigma) c_0^{-\sigma} \mathbf{v}_{PF} e^{-g\sigma t}$$

where $\mathbf{v}_{PF} > \mathbf{0}$ is the right eigenvector of matrix $\mathbf{A}$ corresponding the λ_{PF} normalized by setting the first component equal to 1.

(2nd regime) If $\lambda_{PF}^{-1} - \delta_x(1-\sigma) \le \rho \le a_{11}^{-1} - \delta_x(1-\sigma)$, then there is a cone of dimension ≥1 of optimal steady states solutions of Problem (P) given by

$$c(t) = c_0 e^{gt},\ \mathbf{x}(t) = \mathbf{x}_0 e^{gt},$$

$$\mathbf{s}(t) = \left[\mathbf{s}_0 - \frac{1}{\delta_x - \delta_z}\mathbf{a} - \mathbf{b}\right] e^{gt} + \left[\frac{1}{\delta_x - \delta_z}\mathbf{a} + \mathbf{b}\right] e^{-\delta_z t} \qquad \text{for } \delta_x - \delta_z \neq 0$$

$$\mathbf{s}(t) = \left[\mathbf{s}_0 + t\mathbf{a}\right] e^{-\delta_z t} \qquad \text{for } \delta_x - \delta_z = 0$$

where

$$g = -\delta_x,\ c_0 > 0,\ \mathbf{x}_0^T = c_0 \mathbf{e}_1^T + \mathbf{a}^T,\ \mathbf{s}_0^T = \mathbf{x}_0^T \mathbf{A} + \mathbf{b}^T$$

$$\mathbf{a} \ge \mathbf{0},\ \mathbf{a}^T \mathbf{v}_0 = \mathbf{a}^T \mathbf{A} \mathbf{v}_0 = 0,\ \mathbf{b} \ge \mathbf{0},\ \mathbf{b}^T \mathbf{v}_0 = 0$$

and with supporting prices and rentals given by

$$\mathbf{v} = c_0^{-\sigma} \mathbf{v}_0 e^{-g\sigma t},\ \mathbf{q} = (\rho + \delta_z + g\sigma) c_0^{-\sigma} \mathbf{v}_0 e^{-g\sigma t}$$

where $\mathbf{v}_0$ is a solution to the system

$$\mathbf{e}_1^T \mathbf{v}_0 = c_0^{-\sigma}$$

$$[\mathbf{I} - (\rho + \delta_x + g\sigma)\mathbf{A}]\mathbf{v}_0 \le \mathbf{0}$$

$$\mathbf{e}_1^T [\mathbf{I} - (\rho + \delta_x + g\sigma)\mathbf{A}]\mathbf{v}_0 = \mathbf{0}$$

$$\mathbf{v}_0 \geq \mathbf{0}.$$

(3rd regime) If $\rho > a_{11}^{-1} - \delta_x(1-\sigma)$, then there is a cone of dimension *n* of optimal steady states solutions of Problem (P) given by

$$c(t) = c_0 e^{gt},\ \mathbf{x}(t) = \mathbf{x}_0 e^{gt},$$

$$\mathbf{s}(t) = \frac{1}{g+\delta_z}\left\{(\delta_z - \delta_x)\mathbf{s}_0 + \left[\left(\mathbf{x}_0^T\mathbf{e}_1\right) - c_0\right]\mathbf{e}_1^T + \mathbf{a} - (\delta_z - \delta_x)\mathbf{b}\right\}e^{gt} +$$

$$+\frac{1}{g+\delta_z}\left\{(g+\delta_x)\mathbf{s}_0 - \left[\left(\mathbf{x}_0^T\mathbf{e}_1\right) - c_0\right]\mathbf{e}_1^T + \mathbf{a} - (\delta_z - \delta_x)\mathbf{b}\right\}e^{-\delta_z t} \qquad \text{for } g+\delta_z \neq 0$$

$$\mathbf{s}(t) = \left\{\mathbf{s}_0 + t\left[\left(\mathbf{x}_0^T\mathbf{e}_1 - c_0\right)\mathbf{e}_1^T + \mathbf{a} + (\delta_x - \delta_z)(\mathbf{b} - \mathbf{s}_0)\right]\right\}e^{gt} \qquad \text{for } g+\delta_z = 0$$

where

$$g = \tfrac{1}{\sigma}(a_{11}^{-1} - \delta_x - \rho) \in (-\infty, -\delta_x),\ c_0 > 0,\ \mathbf{x}_0 = \frac{\sigma c_0}{\rho a_{11} - (1-\delta_x a_{11})(1-\sigma)}\mathbf{e}_1 + \mathbf{a},\ \mathbf{s}_0^T = \mathbf{x}_0^T\mathbf{A} + \mathbf{b}^T$$

$$\mathbf{a} \geq \mathbf{0},\ \mathbf{a}^T\mathbf{v}_0 = \mathbf{a}^T\mathbf{A}\mathbf{v}_0 = 0,\ \mathbf{b} \geq \mathbf{0},\ \mathbf{b}^T\mathbf{v}_0 = 0$$

and with supporting prices and rentals given by

$$\mathbf{v} = c_0^{-\sigma}\mathbf{v}_0 e^{-g\sigma t},\ \mathbf{q} = (\rho + \delta_z + g\sigma)c_0^{-\sigma}\mathbf{v}_0 e^{-g\sigma t}$$

where $\mathbf{v}_0$ is a solution to the system

$$\mathbf{e}_1^T\mathbf{v}_0 = c_0^{-\sigma}$$

$$[\mathbf{I} - a_{11}^{-1}\mathbf{A}]\mathbf{v}_0 \leq \mathbf{0}$$

$$\mathbf{e}_1^T[\mathbf{I} - a_{11}^{-1}\mathbf{A}]\mathbf{v}_0 = \mathbf{0}$$

$$\mathbf{v}_0 \geq \mathbf{0}.$$

The interpretation of the theorem is simple. If $g > -\delta_x$, then all commodities need to be produced and therefore *n* processes are to be operated. Thus, the *n* equations relating prices and rate of profit relative to the operated processes (the no arbitrage conditions) determine both the *n*

- 1 relative prices and the real rate of profit. If $g < -\delta_x$, the only process which is relevant is the process producing commodity 1. Since the inputs used by this process are produced by the process itself at a rate larger than the growth rate, all commodities used in the production of commodity 1 except commodity 1 itself have a zero price. In other words, production is reduced to the production of commodity 1 by means of commodity 1 and free goods. Hence, similarly to the previous case the equation relating prices and the rate of profit relative to the operated process producing commodity 1 can determine the rate of profit (apart from commodity 1, all commodities which are either produced or conserved have a zero price). If $g = -\delta_x$, then once again the only relevant process is that producing commodity 1 and the inputs used by this process are produced by the process itself, but this is realized at a rate equal to the growth rate and therefore these commodities (except commodity 1) may have either a positive or a zero price. Those with a positive price cannot be separately produced or conserved; their existing stocks can be regarded as stocks of 'renewable' resources for which a growth rate of $-\delta_x$ can be granted in the production of commodity 1.

4. A comparison with the von Neumann-Sraffa-Morishima models

Let us consider the case of $g + \delta_z \geq 0$, taking account of the fact that in steady state equation (4) holds. Let us substitute r for $\rho + g\sigma$ in inequality (3.2) and in equation (3.3). Let us drop Assumptions [HP3], [HP4], [HP4.1], [HP4.2] in order to allow for choice of techniques and joint production and, therefore, substitute matrix **B** for **I** in inequalities (3.2), (3.7), (3.8) and in equations (3.3) and (3.6). Then we obtain from inequalities (3.2), (3.7), (3.9) and equations (3.3) and (3.6) exactly the model analyzed in the von Neumann-Sraffa-Morishima literature developed in the sixties and seventies, with contributions up to now (see the treatise by Kurz and Salvadori, 1995, summarizing the whole approach):

$$[\mathbf{B} - (r + \delta_x)\mathbf{A}]\mathbf{v}_0 \leq \mathbf{0} \tag{6.1}$$

$$\mathbf{x}_0^T[\mathbf{B} - (r + \delta_x)\mathbf{A}]\mathbf{v}_0 = \mathbf{0} \tag{6.2}$$

$$\mathbf{x}_0^T[\mathbf{B} - (g + \delta_x)\mathbf{A}]\mathbf{v}_0 = c_0\mathbf{e}_1^T\mathbf{v}_0 \tag{6.3}$$

$$\mathbf{x}_0^T[\mathbf{B} - (g + \delta_x)\mathbf{A}] \geq c_0\mathbf{e}_1^T \tag{6.4}$$

$$c_0 > \mathbf{0},\ \mathbf{v}_0 \geq \mathbf{0},\ \mathbf{x}_0 \geq \mathbf{0}. \tag{6.6}$$

The main difference consists in the fact that in that literature there was almost always at least one primary factor called "labor". In the cases in which no primary factor was taken into consideration, or the wage of laborers was taken to be zero, an extra inequality was mentioned:

$$\mathbf{x}_0^T\mathbf{B}\mathbf{v}_0 > 0. \qquad (6.7)$$

In our case it is certainly satisfied since equation (3.1) and inequality (3.9) hold.

A first problem to be analyzed is the following. The first regime mentioned in the Classification Theorem is certainly mentioned in that literature. Actually it is "the" result that that literature would have predicted for an economy with stationary relative prices and a wage rate equal to zero. But what about the other regimes? The von Neumann-Sraffa-Morishima literature has rarely considered negative growth rates. Could one expect such a difference?

The problem arises since for $g < -\delta_x$ the distinction between single production and joint production is not relevant. This section is devoted to clarify this point as a contribution to the understanding of the problems that multisector models imply. A graphical exposition in terms of two goods would be here sufficient. Consider a joint production process

$$\mathbf{a} \rightarrow \mathbf{b},$$

and let us represent it in a commodity space in which appear vectors **a** and **b** and vector $\mathbf{b}-(g+\delta_x)\mathbf{a}$. In figure 2 we have drawn four alternative vectors $\mathbf{b}-(g+\delta_x)\mathbf{a}$, depending on the size of g: $g' < -\delta_x < g'' < g'''$. In figure 3, instead, there are two single production processes, one producing commodity 1 and one producing commodity 2. Also in this case we have drawn in the figure alternative vectors $\mathbf{b}_i-(g+\delta_x)\mathbf{a}_i$. It is immediately recognized that when $g+\delta_x$ is positive the cone consisting of convex combinations of vectors $\mathbf{b}_i-(g+\delta_x)\mathbf{a}_i$ (i = 1, 2) includes the positive orthant, whereas for negative values of $g+\delta_x$ the same cone is included in the positive orthant. Both processes concur to produce the consumption vector at the required growth rate only if the consumption vector is *internal* to that cone. Let us pay attention to the other two cases: the one in which the consumption vector is *external* to the cone and the one in which the consumption vector is *on the boundary* of the cone.

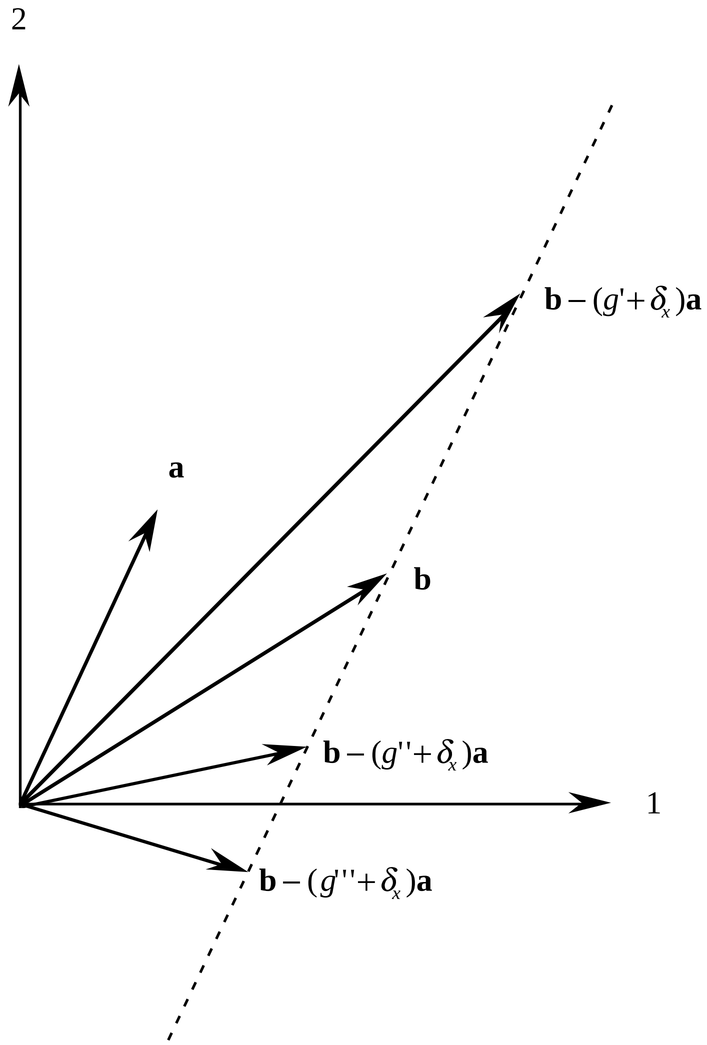


Figure 2

In a two commodity economy the fact that the consumption vector is external to the above mentioned cone means that no convex combination of processes can supply an amount of the required consumption without overproduction of a commodity, which, as a consequence, has a zero price. Moreover, there is a problem of choice of technique even if the original model contemplated a number of processes equal to the number of commodities: as a matter of fact the model is equivalent to one in which the commodity with a price equal to zero does not exist, whereas the number of processes is not changed. The same argument is immediately applicable to an *n*-commodity economy: once again some price need to be zero, a convex combination of the operated processes meets the components of the consumption vector with a positive price and overproduction of commodities with a zero price.

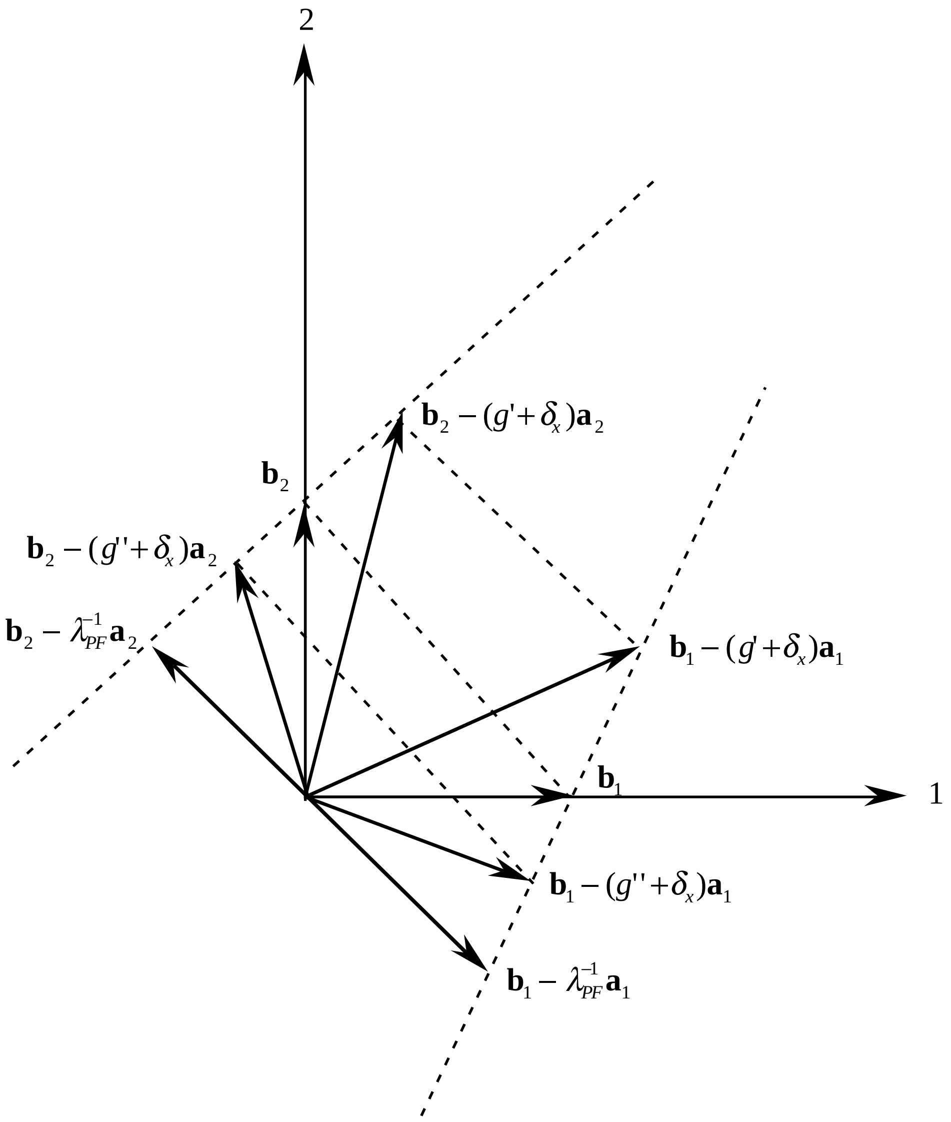


Figure 3

In a two commodity economy the fact that the consumption vector is on the boundary of the cone means that there is a convex combination of processes which can supply an amount of the required consumption without the condition that a commodity is overproduced, but this convex combination is actually made up by one process only: no commodity need to be overproduced, but the no arbitrage conditions determine a number of constraints lower than the number of the prices to be ascertained. Hence we have to study if the fact that the consumption vector is on the boundary of the cone is just a case or there are economic forces at work to impose this condition. In the former case prices are undetermined and vary in a range. In the latter case the forces at work have an impact on prices which can be determined. The same argument is immediately applicable to an *n*-commodity economy: once again the no arbitrage conditions determine only a number of relations among prices which are not enough to determine prices. Then the economic forces which push the consumption vector to be on the boundary may determine the further constraints which are able to determine prices, otherwise prices are not fully determined and may vary in a range.

With these arguments in mind we can go to analyze the last two regimes mentioned in the Classification Theorem. The third one is clearly a case in which the consumption vector, which in our case is proportional to the first unit vector, is outside the cone: all commodities needed for the production of commodity 1 except commodity 1 itself are overproduced. The second regime mentioned in the Classification Theorem is clearly a case in which the consumption vector is on the boundary of the cone. In this range the no arbitrage conditions are not able to fully determine prices; then in the long run an increment in ρ pushes down the prices of the inputs of commodity 1 pushing the rate of profit *r* up in such a way that the increment in ρ is exactly compensated and the growth rate is unchanged. Note that if the growth rate would be pushed up, then the no arbitrage conditions would impose prices proportional to $\mathbf{v}_{PF}$, whereas if it would be pushed down, then the no arbitrage conditions would impose zero prices for all inputs of commodity 1 apart commodity 1 itself.

The arguments developed in this section suggest that

- if there is joint production the second and the third regime of the Classification Theorem do not need to be connected with negative values of the growth rate,
- if more than one commodity is consumed (in a single production setting), then
  - the first regime of the Classification Theorem may hold also for values of the growth rate lower than $-\delta_x$,
  - if the proportion in which commodities are consumed depends on prices, then the second regime may determine a relationship between the rate of profit and the growth rate which may be different from an horizontal segment,
  - if there is continuous substitution in consumption, the third regime of the Classification Theorem may not exist.

These issues will be dealt with in sections 6 and 7. More in general, the arguments developed in the first part of this section suggest that many of the results obtained in long-period models of Classical inspiration like those of von Neumann and Sraffa can at least partially be imported in the framework here presented. In particular the problem of choice of technique and that of joint production appear to be easily handled. This is not developed in this paper. Similarly, some difficulties recognized in that literature should have corresponding difficulties here. In particular we know that dropping Assumption [HP4.2], or his general form [HP5], may lead to difficulties. The analog of these difficulties in the present framework will be illustrated in next section.

5. On the properties of matrix A

It has been suggested that the AK model "becomes more plausible if we think of K in a broad sense to include human capital" (Barro and Sala-i-Martin, 1995, p. 39). We would like to suggest an alternative interpretation of the multisector version of that model. This interpretation looks at the AK model from the perspective of the classical economists from Adam Smith to David Ricardo. Let $\hat{\mathbf{A}}$ be the usual material input matrix used in input-output analysis and let $\mathbf{l}$ be the input vector of (simple) labor. If we assume that the real wage rate per unit of labor is defined by the vector $\mathbf{b}$, then the matrix $\mathbf{A}$ above can be seen as $\mathbf{A} = \hat{\mathbf{A}} + \mathbf{l}\mathbf{b}^T$. In this interpretation the AK model takes the real wage rate as exogenously given and determines the rate of growth on the basis of this given: a larger real wage rate implies a larger λ_{PF} and, as a consequence, a lower rate of profits and a lower rate of growth.

A formal consequence of this interpretation is that if labor enters directly into the production of all commodities, to each positive element of vector $\mathbf{b}$ corresponds a positive column of matrix $\mathbf{A}$. This means that there exists at least one "basic commodity", to use an expression used by Sraffa.[4] This eliminates some silly cases. These cases were excluded in our previous paper since matrix $\mathbf{A}$ was assumed to be indecomposable; hence all commodities were assumed to be

4 A basic commodity is a commodity entering directly *or* indirectly into the production of all commodities. In this case there is a commodity entering *directly* into the production of all commodities.

basic. It is interesting, however, to check what happens when there is a basic commodity, but not all commodities are basic. The literature on von Neumann-Sraffa-Morishima models has recognized that there are difficulties when a non-basic enters directly or indirectly into its own production. In this section we will evaluate an example of this type in the framework here presented. Before then, however, let us consider another aspect of the suggested interpretation of matrix **A**.

Smith distinguished consumption goods in necessaries, conveniences, and luxuries. Workers were supposed to consume necessaries and some conveniences, whereas capitalists and mainly landlords consumed also luxuries. If this suggestion is to be taken into consideration, the assumption of a single consumption good is not tenable, since the consumer is actually a "capitalist" (the workers' consumption being subsumed in the elements of matrix **A**). If, for the sake of simplicity, the assumption that commodity 1 is the only consumable is maintained, we have to distinguish whether commodity 1 a necessary, a convenience, or a luxury good. In the first case it can be presumed that the first column of matrix **A** is positive, whereas in the third case it can be presumed that a column of matrix **A** is positive, but this column is not the first one. In the first case Assumption [HP7] is perfectly justified and, moreover, the vector **a** mentioned in the second and third regime of the Classification Theorem is necessarily equal to **0**. An example concerning multiplicity of consumption goods will be provided in section 7.

In the following of this section we present an example with two commodities: commodity 2 enters directly into the production of both commodities, whereas commodity 1 enters directly only into its own production. This simple model is analyzed only to illustrate the difficulties that a decomposable matrix **A** can generate, however the case of decomposable matrices is not uncommon in the new growth literature: the model by Lucas (1988), for instance, is of this type (for a similar remark, see also McKenzie, 1998).

Let (P) be the problem

$$V(\mathbf{s}^*) = \sup \int_0^\infty e^{-\rho t} \frac{c^{1-\sigma}}{1-\sigma} dt$$

$$\dot{\mathbf{s}}^T = \mathbf{x}^T \mathbf{I} - \delta_x \mathbf{x}^T \mathbf{A} - \delta_z (\mathbf{s}^T - \mathbf{x}^T \mathbf{A}) - \mathbf{c}^T$$

$$\mathbf{x}^T \mathbf{A} \leq \mathbf{s}^T, \mathbf{x} \geq \mathbf{0}, \mathbf{c} \geq \mathbf{0}$$

$$\mathbf{s}(0) = \mathbf{s}^* > \mathbf{0} \quad given.$$

where $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}$ and $a_{11} < a_{22}$. It is easily checked that the steady state optimal solutions can be represented in four regimes which are depicted in figure 4. Three regimes are the same mentioned in the Classification Theorem. The fourth regime is characterized by

$$r = a_{11}^{-1} - \delta_x,$$

$$-\delta_x < g < a_{22}^{-1} - \delta_x,$$

$$a_{11}^{-1} - a_{22}^{-1}\sigma - \delta_x(1-\sigma) < \rho < a_{11}^{-1} - \delta_x(1-\sigma).$$

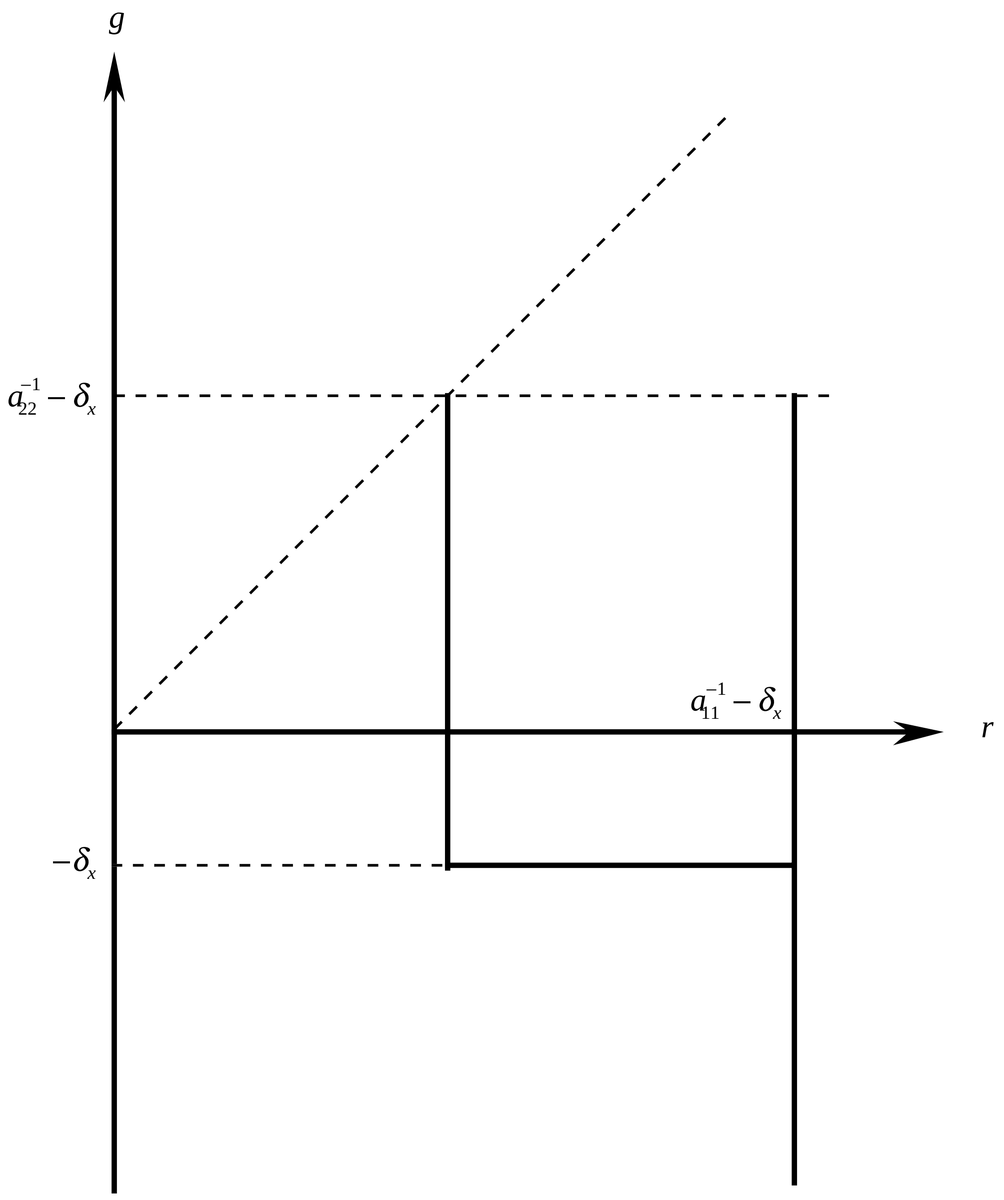


Figure 4

In this regime commodity 2 is (over)produced and its price equals 0. The growth rate cannot be equal or larger than $a_{22}^{-1} - \delta_x$ because otherwise the intensity of operation of the process producing commodity 1 would be nought or negative (and therefore consumption would be nought or negative). Hence if $(a_{22}^{-1} - \delta_x)(1-\sigma) < \rho \leq a_{11}^{-1} - a_{22}^{-1}\sigma - \delta_x(1-\sigma)$, then there is only the first regime of the Classification Theorem. If $a_{11}^{-1} - a_{22}^{-1}\sigma - \delta_x(1-\sigma) < \rho < a_{11}^{-1} - \delta_x(1-\sigma)$, then there are two possible results: either the first or the second regime of the Classification Theorem (but never both) and the fourth here mentioned. For larger values of ρ the results are not different from those mentioned in the Classification Theorem.

It is the case to mention a possible interpretation of the fourth regime. Commodity 2 behaves like a renewable resource which, if it is left to itself, grows at rate $a_{22}^{-1} - \delta_x$. Producers peak up it, in order to produce commodity 1, as fruits were collected in the Eden: at no cost.

6. Two examples with joint production

In this section we present two examples involving joint production. The first example shows that the second regime of the Classification Theorem and part of the third can actually occur for positive growth rates. The second example shows that the first regime does not need to exist when joint production is involved.

Let (P) be the problem

$$V(\mathbf{s}^*) = \sup \int_0^\infty e^{-\rho t} \frac{c^{1-\sigma}}{1-\sigma} dt$$
$$\dot{\mathbf{s}}^T = \mathbf{x}^T\mathbf{B} - \delta_x \mathbf{x}^T\mathbf{A} - \delta_z(\mathbf{s}^T - \mathbf{x}^T\mathbf{A}) - \mathbf{c}^T$$
$$\mathbf{x}^T\mathbf{A} \leq \mathbf{s}^T, \mathbf{x} \geq \mathbf{0}, \mathbf{c} \geq \mathbf{0}$$
$$\mathbf{s}(0) = \mathbf{s}^* > \mathbf{0} \quad given.$$

where $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. It is easily checked that the steady state optimal solutions can be represented in the same three regimes referred to in the Classification Theorem. The first regime is characterized by

$$r = \sqrt{3} - \delta_x,$$

$$1 - \delta_x < g < \sqrt{3} - \delta_x,$$

$$(\sqrt{3}-\delta_x)(1-\sigma) < \rho < \sqrt{3}-\sigma-\delta_x(1-\sigma).$$

The second regime is characterized by

$$g = 1-\delta_x,$$

$$\sqrt{3}-\delta_x \le r \le 2-\delta_x,$$

$$\sqrt{3}-\sigma-\delta_x(1-\sigma) \le \rho \le \sqrt{3}-\sigma-\delta_x(1-\sigma).$$

In the third regime $g < 1-\delta_x$, $r = 2-\delta_x$, $\rho > \sqrt{3}-\sigma-\delta_x(1-\sigma)$. Hence if $\delta_x < 1$, the second regime of the Classification Theorem and part of the third occur for positive values of the growth rate.

Let (P) be the same problem above, except that the input- output-matrices are now given by $\mathbf{A} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$. It is easily checked that the steady state optimal solutions can be represented by two of the three regimes referred to in the Classification Theorem. The second process, in fact, is inefficient with respect to the first process and therefore is never operated. Hence the first regime of the Classification Theorem cannot exist. The second regime is characterized by $g = 4-\delta_x$, $4-\delta_x < r \le 8-\delta_x$, $(4-\delta_x)(1-\sigma) < \rho \le 8-4\sigma-\delta_x(1-\sigma)$. Let us remark that the point on the 45° line is not a solution, and therefore does not belong to the second regime, because the price of commodity 1 would be 0 and therefore consumption would be infinite. In all previous examples the point on the 45° line was not a solution, and therefore did not belong to the first regime, because otherwise the consumption would have been nought. The third regime is characterized by $g < 4-\delta_x$, $r = 8-\delta_x$, $\rho > 8-4\sigma-\delta_x(1-\sigma)$.

7. An example with multiple consumption commodities

In this section we present an example involving two consumption commodities. First of all we have to notice that we cannot use the theorems stated in Section 3 above. Nevertheless an evaluation of an example can be attempt by a straitforward generalization of Theorem 1 on sufficient conditions and taking account of a partition of all possible cases.

Let (P) be the problem

$$V(\mathbf{s}^*) = \sup \int_0^\infty e^{-\rho t} \log(c_1^{1-\alpha} c_2^{\alpha}) dt$$
$$\dot{\mathbf{s}} = \mathbf{x} - \mathbf{s} - \mathbf{c}$$
$$\mathbf{x}^T \mathbf{A} \le \mathbf{s}^T$$
$$\mathbf{x} \ge \mathbf{0}, \mathbf{c} \ge \mathbf{0}$$
$$\mathbf{s}(0) = \mathbf{s}^* \ge \mathbf{0} \quad given.$$

where $\mathbf{A} = \begin{bmatrix} \frac{1}{8} & \frac{1}{4} \\ \frac{1}{8} & 0 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. If α = 0 or α = 1, there is a single consumption commodity. Hence, let 0 < α < 1. Since the instantaneous Cobb-Douglas utility function determines consumption share, in value, constant and depending only on α, we have that along any optimal path:

$$\frac{c_1 v_1}{c_2 v_2} = \frac{1-\alpha}{\alpha} .$$

Therefore, along any optimal path both prices and both consumption levels are positive. Hence in a steady state optimal solution

$$c_1 = x_1 - (1+g)s_1 > 0, c_2 = x_2 - (1+g)s_2 > 0$$

$$g\mathbf{s} = \mathbf{x} - \mathbf{s} - \mathbf{c}$$

$$\mathbf{x}^T \mathbf{A} = \mathbf{s}^T$$

Let us partition all possible cases on the basis of operated processes, that is

(i) both processes are operated

(ii) only the first process is operated

(iii) only the second process is operated.

If both processes are operated, then r = 3, $\frac{v_1}{v_2} = 2$, $\frac{c_1}{c_2} = \frac{1-\alpha}{2\alpha}$, $x_1 = \frac{8(4-3\alpha+\alpha g)}{(1-\alpha)(27-6g-g^2)} c_1$, $x_2 = \frac{1+6\alpha+g-2\alpha g}{(1-\alpha)(27-6g-g^2)} c_1$. There are two critical values for the parameter α: 1/3, and 1/2. For 0 < α < 1/3 vector **x** is positive if and only if $g < \frac{3\alpha-4}{\alpha}$ or $\frac{1+6\alpha}{2\alpha-1} < g < 3$. For α = 1/3 vector **x** is positive for g < 3. For 1/3 < α < 1/2 vector **x** is positive if and only if $g < \frac{1+6\alpha}{2\alpha-1}$ or

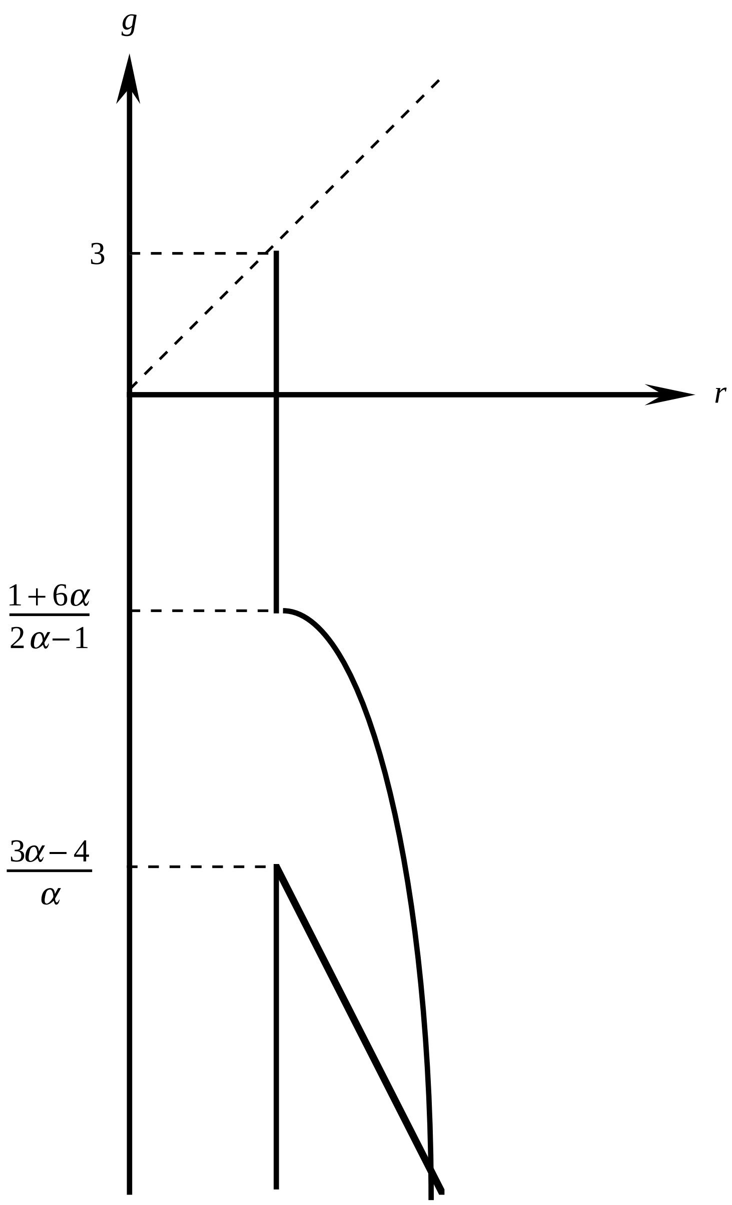


Figure 5a

$\frac{3\alpha-4}{\alpha} < g < 3$. For 1/2 ≤ α < 1 vector **x** is positive if and only if $\frac{3\alpha-4}{\alpha} < g < 3$. If only the first process is operated, then $\frac{v_1}{v_2} = \frac{2(1+r)}{7-r}$, $\frac{c_1}{c_2} = \frac{g-7}{2(1+g)}$, 3 < r < 7, and g < -1. Hence $g = \frac{7+(8\alpha-1)r}{(8\alpha-7)+r}$ and α < 1/2. If only the second process is operated, then $\frac{v_1}{v_2} = \frac{8}{1+r}$, $\frac{c_1}{c_2} = -\frac{(1+g)}{8}$, r > 3, and g ≤ -1. Hence $g = -\frac{1+(1-\alpha)r}{\alpha}$. Figures 5 provide the relationship

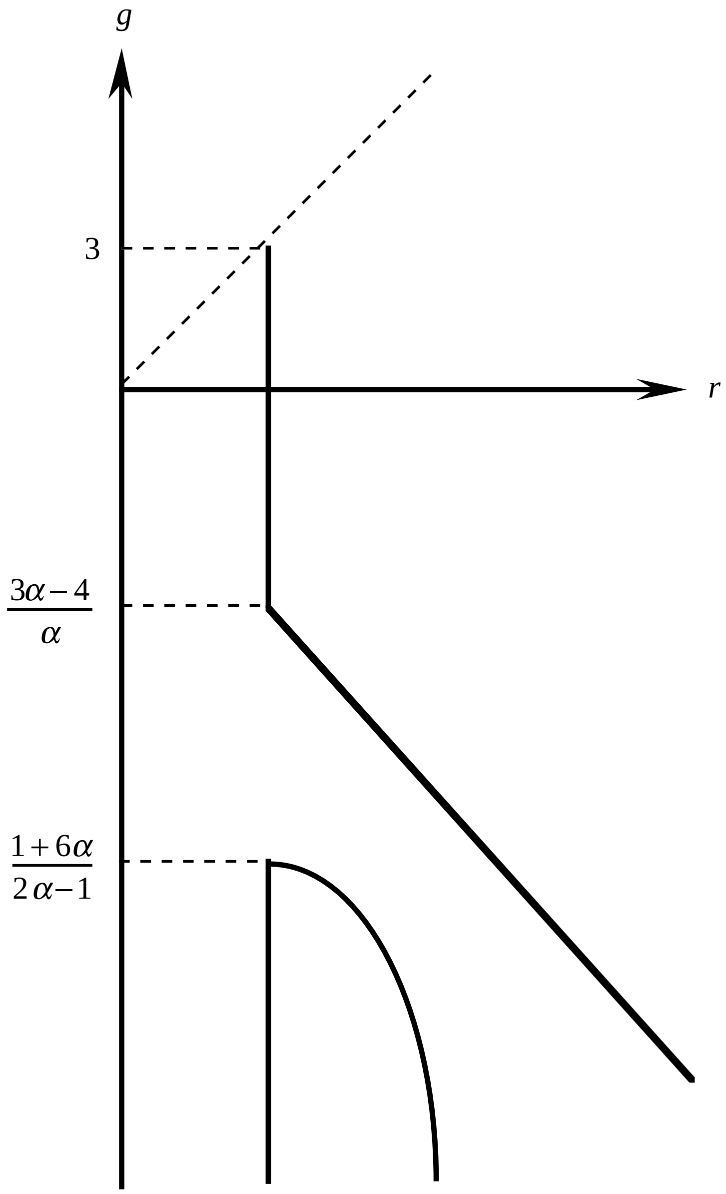
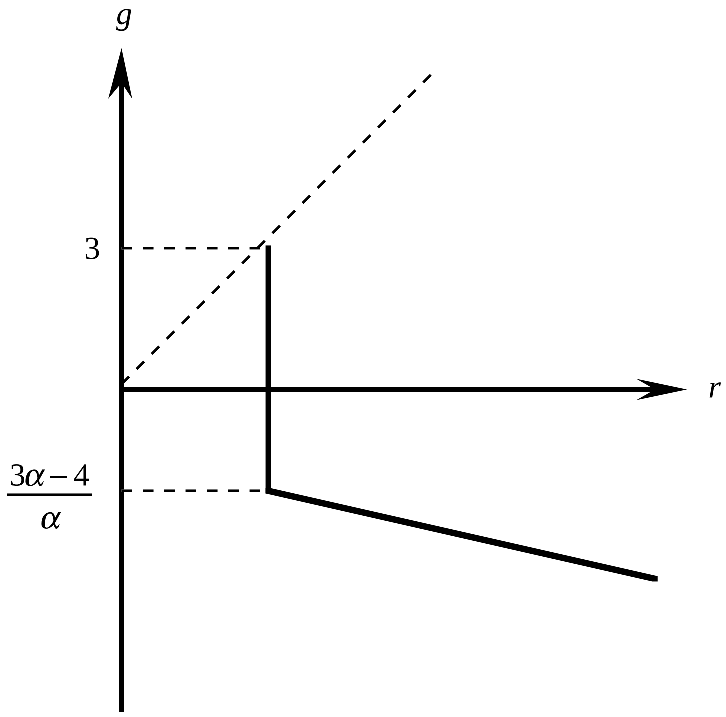


Figure 5b

between g and r for α's in one of the three relevant ranges: $0 < \alpha < 1/3$, $1/3 < \alpha < 1/2$, $1/2 \leq \alpha < 1$, respectively.

8. Concluding remarks

This paper has investigated a number of problems which are absent in any single sector economy, but can be present in a multiple sector economy. This has been done with the help of a generalization of the multisector 'AK model' in continuous time we analyzed in a previous



Figures 5c

paper. This analysis has also shown how this model is connected to the von Neumann-Sraffa-Morishima linear models investigated in the sixties and seventies.

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