Technical change, effective demand and economic growth

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Abstract

The model developed in this paper has distinctly classical, but also Schumpeterian and Keynesian features. The main analysis is explored in an aggregate (one goods) setting, but many of the results carry over to a multi-sectoral setting. Sections 2 and 3 present the main components of the model in its equilibrium setting. Here we show the classical wage-profits and consumption vs. investment/growth trade-offs. We also show the conditions which have to be satisfied for full employment growth. Important are distributive requirements - particularly the determination of the real wage - which allow the economy to achieve full employment in the face of (positive) productivity shocks and changes in the available labour force. This issue is further explored in section 4 in the context of changing (exogenous) growth rates of productivity and labour force growth. Real wage adjustments are also required if the economy is to remain on a full employment path and profit-receivers change their behaviour with respect to investment vs. consumption spending. Section 5 moves to discuss off-steady state analysis and introduces further components into the model. In 'disequilibrium' there is no immediate price-to-cost adjustment; thus Schumpeterian rents emerge and we allow for some bargaining between workers and capitalists over these rents. In the wage equation features also the impact of the unemployment rate. Hence, as the economy absorbs a (positive) productivity shock, there is a shift of distributive shares towards 'rents', and different distributive scenarios of such rents can emerge. The next important ingredient to understand the growth implications is to look at the 'spending' pattern out of these rents: these can go into consumption spending, investment spending or they 'leak out' of the real system (e.g. into liquid or financial assets). We show that the classical results obtained earlier with regard to wage-profit consumption-growth trade-offs can be seriously modified when such 'off-equilibrium' dynamics are explored. Here, both Schumpeterian and Keynesian features emerge. Section 6 (still incomplete) attempts to model the interaction between the financial and the real sector of the economy. One of the motivations behind the study is to arrive at an understanding of the so-called 'Solow Paradox', i.e. that there might be evidence of a strong positive technology boost which, however, does not translate - at least for some time - into higher economic growth.

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TECHNICAL CHANGE, EFFECTIVE DEMAND AND ECONOMIC GROWTH

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1 Introduction

The model developed in this paper has distinctly classical, but also Schumpeterian and Keynesian features. The main analysis is explored in an aggregate (one goods) setting, but many of the results carry over to a multi-sectoral setting (for the latter see Landesmann and Stehrer, 2000, and Stehrer, 2001). Sections 2 and 3 present the main components of the model in its equilibrium setting. Here we show the classical wage-profits and consumption vs. investment/growth trade-offs. We also show the conditions which have to be satisfied for full employment growth. Important are distributive requirements - particularly the determination of the real wage - which allow the economy to achieve full employment in the face of (positive) productivity shocks and changes in the available labour force. This issue is further explored in section 4 in the context of changing (exogenous) growth rates of productivity and labour force growth. Real wage adjustments are also required if the economy is to remain on a full employment path and profit-receivers change their behaviour with respect to investment vs. consumption spending. Section 5 moves to discuss off-steady state analysis and introduces further components into the model. In 'disequilibrium' there is no immediate price-to-cost adjustment; thus Schumpeterian rents emerge and we allow for some bargaining between workers and capitalists over these rents. In the wage equation features also the impact of the unemployment rate. Hence, as the economy absorbs a (positive) productivity shock, there is a shift of distributive shares towards 'rents', and different distributive scenarios of such rents can emerge. The next important ingredient to understand the growth implications is to look at the 'spending' pattern out of these rents: these can go into consumption spending, investment spending or they 'leak out' of the real system (e.g. into liquid or financial assets). We show that the classical results obtained earlier with regard to wage-profit consumption-growth trade-offs can be seriously modified when such 'off-equilibrium' dynamics are explored. Here, both Schumpeterian and Keynesian features emerge. Section 6 (still incomplete) attempts to model the interaction between the financial and the real sector of the economy. One of the motivations behind the study is to arrive at an understanding of the so-called 'Solow Paradox', i.e. that there might be evidence of a strong positive technology boost which, however, does not translate - at least for some time - into higher economic growth.

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2 The model

In the main text we explore a one good model. Most results can be extended to the multisectoral case (see appendix). In this section we present the model.

2.1 Technology

Technology is denoted by a pair of input coefficients $a(t)$ and $b(t)$ where $a(t)$ denotes the input coefficient of intermediate goods and $b(t)$ the input coefficient for labour. The standard relationships are:

1. aq is the intermediate demand for production of output q
2. $a^{-1}j$ gives the output with input j
3. $(1-a)q$ is the amount of output q not used as intermediate input
4. $(1-a)^{-1}f$ gives the necessary gross production for a final demand f

The labour input coefficient $b(t)$ is assumed to be larger than zero (thus we assume that some labour is always used in production). Labour productivity is then $b^{-1}(t)$.

2.2 Wages and unit labour costs

The (nominal) wage rate is denoted by

$$w(t)$$

Unit labour costs are defined as

$$b(t)w(t)$$

2.3 Costs and prices

Unit costs are defined as

$$c(t) = p(t)a(t) + w(t)b(t)$$

Prices are

$$p(t) = (1+\pi)c(t)$$

This formulation implies that wages are paid at the beginning of the period, i.e. *ante factum.*[2]

[2]The results for prices paid *post factum*,

$$p(t) = (1+\pi)p(t)a(t) + w(t)b(t)$$

are derived in the appendix.

The equilibrium of the price system can easily be determined in the case that the coefficient a, the unit labour costs wb and thus w and b, and the mark-up π are exogenously given and constant. The equilibrium price is

$$p = (1+\pi)wb\,[1-(1+\pi)a]^{-1}$$

Given the price one may express the wage rate as

$$w = \frac{p\,[1-(1+\pi)a]}{(1+\pi)b}$$

The real wage in equilibrium is

$$\frac{w}{p} = \frac{[1-(1+\pi)a(t)]}{(1+\pi)b(t)}$$

The ratio of unit labour costs to prices is

$$\frac{w}{p}b = \frac{1-(1+\pi)a}{1+\pi}$$

In the special case where $\frac{w}{p} = 0$ we obtain the maximum mark-up rate

$$\pi^* = \frac{1}{a} - 1$$

2.4 Profits and rents

The per unit (equilibrium) profit is defined as a mark-up on costs

$$r(t) = \pi c(t) = \pi\big(p(t)a(t) + w(t)b(t)\big)$$

In the case that the price is not at the equilibrium level there arise per unit-rents which are defined as:

$$s(t) = p(t) - (1+\pi)c(t) = p(t)\,[1-(1+\pi)a(t)] - (1+\pi)w(t)b(t)$$

In equilibrium these rents are zero as one can show by inserting the equilibrium price

$$\begin{aligned} s &= (1+\pi)wb\,[1-(1+\pi)a]^{-1}\,[1-(1+\pi)a] - (1+\pi)wb \\ &= (1+\pi)wb - (1+\pi)wb \\ &= 0 \end{aligned}$$

In the following we define a scalar $m(t)$ which adds up profits $r(t)$ and rents $s(t)$, which we refer to as 'total profits',

$$m(t) = r(t) + s(t) = p(t) - c(t) = p(t)(1-a(t)) - wb(t)$$

In equilibrium total profits per unit in real terms are

$$\begin{aligned} \frac{m}{p} = \frac{r}{p} &= (1-a) - \frac{w}{p}b \\ &= (1-a) - \frac{[1-(1+\pi)a]}{(1+\pi)} \\ &= \frac{\pi}{1+\pi} \end{aligned}$$

2.5 Demand components and total demand

Next we turn to the quantity system. The quantities are denoted by

$$q(t)$$

Demand consists of demand for three different components:

1. Intermediate goods
2. Consumption goods
3. Investment goods

2.5.1 Demand for intermediate products

First there is demand for intermediate goods used in production,

$$a(t)q(t)$$

Intermediate goods can be interpreted as capital stock, which in this model consists only of circulating capital.

2.5.2 Wage income

Workers earn a nominal wage rate $w(t)$ and thus total nominal wage income is $w(t)b(t)q(t)$ as $b(t)q(t)$ is the labour demand. In real terms this is

$$f_w(t) = \frac{w(t)}{p(t)} b(t)q(t) = \frac{w(t)}{p(t)} l(t)$$

With constant b, w, π, a, p demand out of wage income depends only on the equilibrium real wage and the quantity of labour demanded

$$f_w(t) = \frac{w}{p} bq(t) = \frac{w}{p} l(t)$$

We shall henceforth assume in classical fashion that all labour income is spent on consumption.[3] By this assumption, $f_w(t)$ also denotes the amount of goods demanded by workers.

[3]The model may also be generalised in the sense that workers can also save. But this is postponed until we introduce a fincancial sector into the model.

2.5.3 Income out of rents

The nominal total profit income is $m(t)q(t)$. Expressed in real terms this is

$$\begin{aligned}
f_m(t) &= \frac{m(t)}{p(t)} q(t) \\
&= \frac{p(t)(1 - a(t)) - w(t)b(t)}{p(t)} q(t) \\
&= (1 - a(t))\, q(t) - \frac{w(t)}{p(t)} b(t) q(t) \\
&= (1 - a(t))\, q(t) - \frac{w(t)}{p(t)} l(t) \\
&= \left((1 - a(t)) - \frac{w(t)}{p(t)} b(t) \right) q(t)
\end{aligned}$$

In fact, this is the residual of total output $q(t)$ minus demand for intermediate inputs and real wage income.

In the case that $m(t) = 0$ one gets

$$(1 - a(t)) = \frac{w(t)}{p(t)} b(t)$$

which means that the workers in the economy reach the maximum amount of consumption.

2.5.4 Total demand

If all incomes are spent then total demand $d(t)$ is the sum of the three components

$$d(t) = a(t)q(t) + f_m(t) + f_w(t)$$

In equilibrium total demand equals total supply

$$\begin{aligned}
q(t) &= a(t)q(t) + f_m(t) + f_w(t) \\
&= a(t)q(t) + \left[(1 - a(t)) - \frac{w(t)}{p(t)} b(t) \right] q(t) + \frac{w(t)}{p(t)} b(t) q(t) \\
&= a(t)q(t) + \left[(1 - a(t)) - \frac{w(t)}{p(t)} b(t) \right] q(t) + \frac{w(t)}{p(t)} b(t) q(t)
\end{aligned}$$

The shares of income and types of demand (intermediate, demand from profit receivers and demand from workers) in real terms are given by

$$1 = a(t) + \left[(1 - a) - \frac{w(t)}{p(t)} b(t) \right] + \frac{w(t)}{p(t)} b(t)$$

Before presenting the solutions to this model we discuss labour demand and supply.

2.6 Labour demand and supply

As already mentioned above, labour demand is

$$l(t) = b(t)q(t)$$

Labour supply is denoted by

$$k(t)$$

We define the unemployment rate as

$$u(t) = \frac{k(t) - l(t)}{k(t)}$$

3 Equilibrium solutions

3.1 Existence

In equilibrium total supply must equal total demand, $q(t) = d(t)$. The condition stated above may be rewritten as

$$q(t) = d(t) = \left(a(t) + \frac{m(t)}{p(t)} + \frac{w(t)b(t)}{p(t)} \right) q(t)$$

Rearranging this condition yields

$$\left((a(t) - 1) + \frac{m(t)}{p(t)} + \frac{w(t)}{p(t)} b(t) \right) q(t) = 0$$

In the case that all income is spent this condition holds at each point in time t (i.e. even in the case that the price system is not in equilibrium) as

$$\begin{aligned} 0 &= (a(t) - 1) + \frac{w(t)}{p(t)} b(t) + \frac{p(t)(1 - a(t)) - w(t)}{p(t)} b(t) \\ &= (a(t) - 1) + \frac{w(t)}{p(t)} b(t) + (1 - a(t)) - \frac{w(t)}{p(t)} b(t) \end{aligned}$$

It is clear that the level of output cannot be determined by this condition alone.

3.2 Steady-state (classical) growth

In the steady-state we take the input coefficients a and b as constant. Wage rates and prices are at their equilibrium values respectively. Further, rents s are equal to zero and real per unit-profit is $\frac{\pi}{1+\pi}$ as shown above. Further we assume that all incomes are spent and that there is no shortage of labour. The demand for labour which equals supply is given by

$$l(t) = bq(t)$$

The three demand components can be calculated as follows: Intermediate demand is

$$aq(t)$$

Demand of workers is

$$f_w(t) = \frac{w}{p} bq(t) = \frac{w}{p} l(t)$$

And finally, demand out of profits and rents must satisfy

$$f_m(t) = \frac{m}{p} q(t)$$

Assuming that $\frac{m}{p} q(t)$ is fully reinvested the demand and supply for inputs will be growing at a rate

$$\frac{af_m(t)}{aq(t)} = \frac{f_m(t)}{q(t)} = \frac{m}{p} = \frac{\pi}{1+\pi} \equiv \gamma_q$$

In this model with circulating capital this represents the (gross and net) accumulation rate. The growth path of the economy can then be written as[4]

$$\begin{aligned} \dot{q}(t) &= (1+\gamma_q)(1-a)^{-1}\big(f_m(t) + f_w(t)\big) - q(t) \\ &= \left[1-(1+\gamma_q)a\right]^{-1}(1+\gamma_q)\big((1-a)f_m(t) + f_w(t)\big) - q(t) \end{aligned}$$

By using $\big(f_m(t) + f_w(t)\big) = (1-a)$ this yields

$$\frac{\dot{q}(t)}{q(t)} = \gamma_q = \frac{\pi}{1+\pi}$$

This dynamic equation shows that profit receivers finance accumulation out of the current income. If $\pi = 0$ the economy remains stationary (all surplus $(1-a)q(t)$ being consumed). With $\pi = \pi^*$ the economy would grow at its maximum rate $\gamma_q^* = \frac{\pi^*}{1+\pi^*}$.

3.3 Full-employment solution

In the steady-state full employment is guaranteed when labour supply equals demand at the initial point of time

$$k(0) = l(0) = bq(0)$$

and there are no labour supply restrictions in the course of growth, i.e.

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{l}(t)}{l(t)} = \frac{\dot{q}(t)}{q(t)}$$

[4] We can see that to achieve this rate of growth, net accumulation of $\gamma_q(1-a)^{-1}\big(f_m(t) + f_w(t)\big)$ is required. This formulation is analogous to the solution of a dynamic Leontief model with only circulating capital (see e.g. Pasinetti, 1977). The equations also correspond to the more general solution for a multi-sectoral economy (see appendix for the solution of the n-sector case).

For wages paid *post factum* these equations are

$$\begin{aligned} \dot{q}(t) &= (1+\gamma_q)(1-a)^{-1}\big(f_m(t) + f_w(t)\big) - q(t) \\ &= \left[1-(1+\gamma_q)a\right]^{-1}(1+\gamma_q)\big(f_m(t) + f_w(t)\big) - q(t) \end{aligned}$$

3.4 Exogenous technical progress

Let us introduce a growth of labour productivity at a constant exogenously given rate which implies that the input coefficient is decreasing, i.e.

$$\dot{b} = -\gamma_b b$$

Under the assumptions that $\pi = 0$ and that the labour supply is constant $\dot{k} = 0$, using the full-employment assumption[5] implies that

$$\frac{\dot{l}}{l} = \frac{\dot{b}}{b} + \frac{\dot{q}}{q} = 0$$

and thus

$$-\frac{\dot{b}}{b} = \frac{\dot{q}}{q} \qquad \text{or} \qquad \gamma_b = \overline{\gamma}_q$$

This means, that for keeping employment at the initial full-employment level output has to grow with the same rate as labour productivity. To guarantee that output grows at this rate two conditions must be met: First, enough demand has to be created and, second, enough resources have to be reinvested. As shown above, the condition

$$\overline{\gamma}_q = \frac{m(t)}{p(t)}$$

has to be satisfied, but now with γ_q given (rather than π given). Rearranging gives

$$\overline{\gamma}_q = \frac{m(t)}{p(t)} = \frac{p(t) - p(t)a - w(t)b(t)}{p(t)} = (1-a) - \frac{w(t)b(t)}{p(t)}$$

As γ_q and $(1-a)$ are constant, $\frac{w(t)b(t)}{p(t)}$ also has to be constant, thus

$$\begin{aligned} 0 &= \frac{\dot{w}}{w} + \frac{\dot{b}}{b} - \frac{\dot{p}}{p} \\ \gamma_w - \gamma_p &= \gamma_b \end{aligned}$$

This says, that the real wage has to grow with the rate of productivity γ_b. This condition assures that consumption demand rises in line with productivity growth. If it would rise beyond productivity growth there would not be enough profits to allow the required net accumulation to take place to assure full employment. If it would grow below that rate a mismatch between net accumulation (i.e. increase in productive capacity) and final consumption demand would take place.

We can also see that the ratio of unit labour costs to price has to be lower than it would be if the economy would not undergo positive productivity growth. We get

$$\overline{\gamma}_q = (1-a) - \frac{w(t)}{p(t)} b(t)$$

[5] A growing labour force is discussed below.

By rearranging this yields

$$\frac{w(t)}{p(t)}b(t) = (1-a) - \overline{\gamma}_q = (1-a) - \gamma_b$$

This condition assures that enough resources are available for keeping the economy growing at the full employment path.

3.5 Exogenous population growth

Next, we turn to a situation where the labour force is growing at aN exogenously given and constant rate, γ_k. For keeping the economy at the full employment level, this means that labour demand must rise with the rate of labour supply

$$\gamma_k = \gamma_l$$

and thus the growth rate of output must be

$$\overline{\gamma}_q = \gamma_k + \gamma_b$$

Again, as γ_b and $(1-a)$ are assumed to be constant, the term $\frac{w(t)}{p(t)}b(t)$ has to be constant which analogously to the above yields

$$\gamma_w - \gamma_p = \gamma_b$$

i.e. the real wage has to grow at the rate of productivity. But, different from above, the ratio of unit labour costs to price will have to be even lower, as

$$\gamma_k + \gamma_b = (1-a) + \frac{w(t)}{p(t)}b(t)$$

or equivalently

$$\frac{w(t)}{p(t)}b(t) = (1-a) - (\gamma_k + \gamma_b)$$

This implies that the consumption level must be lower to 'save' relatively more output for reinvestment. Output must be growing not only to keep employment at a constant level (thus only capturing the labour saving effects of technical progress) but also to create employment for the growing labour force (population). The necessary balance between demand and the growth of productive capacity is created by increases in the real wage and the demand from new workers.

3.6 Consumption out of profits

So far we have assumed that workers income is spent on consumption and income out of total profits is spent for investment. We now relax this assumption to the case in which profit earners spend part of their income on consumption and the other part on investment. The part of income spent on consumption is denoted by κ_m with $0 \leq \kappa_m \leq 1$.

In the model of steady-state classical growth the demand and supply for input will be growing in this case at rate

$$(1-\kappa_m)\frac{af_m(t)}{aq(t)} = (1-\kappa_m)\frac{f_m(t)}{q(t)} = (1-\kappa_m)\frac{m}{p} = (1-\kappa_m)\frac{\pi}{1+\pi} \equiv (1-\kappa_m)\gamma_q$$

Thus the growth rate of the economy is reduced. The growth path of the economy is then

$$\dot{q}(t) = \big(1+(1-\kappa_m)\gamma_q\big)(1-a)^{-1}\big(\underbrace{(1-\kappa_m)f_m(t)}_{\text{Investment demand}} + \underbrace{\kappa_m f_m(t) + f_w(t)}_{\text{Consumption demand}}\big) - q(t)$$

Also in the case of exogenous technological progress and population growth one has to take into account that part of total profit income is spent on consumption. The condition for full-employment remains

$$\overline{\gamma}_q = \gamma_k + \gamma_b$$

For the economy to provide enough resources to grow at the rate $\overline{\gamma}_q$ the new condition becomes

$$\overline{\gamma}_q = (1-\kappa_m)\frac{m(t)}{p(t)} = (1-\kappa_m)\left((1-a) - \frac{w(t)}{p(t)}b(t)\right)$$

which yields a per unit real wage

$$\frac{wb}{p} = (1-a) - \frac{\gamma_k+\gamma_b}{1-\kappa_m}$$

where

$$(1-a) \geq \frac{\gamma_k+\gamma_b}{1-\kappa_m}$$

has to be satisfied. In fact, this condition says that consumption out of total profits has to be financed by lower real wage income. (In the case that $\kappa_m = 1$ there would be no possibility for the economy to grow whatsever.)

4 Underemployment equilibria

4.1 Stationary economy

If there is no technical progress, no exogenous population growth and $\pi = 0$ the equilibrium of the economy is given by

$$\gamma_q = 0$$

and

$$(1-a) = \frac{w}{p}b$$

The output is allocated to (intermediate) investment and consumption demand so that the economy exactly reproduces itself over time. If $(1-a) < \frac{w}{p}b$ the economy would contract and if $(1-a) > \frac{w}{p}b$ the economy would be able to grow.

What happens if the economy starts in an unemployment situation $u(t) > 0$. The answer is quite simple: nothing, if the factum of unemployment would have no impact on the other variables. One way out of the unemployment situation will be (this is shown below in the disequilibrium dynamic model), that the (real) wage rate is falling, rents are emerging, and these rents enable the economy to grow until it reaches the full-employment equilibrium. The same situation is reached if prices are growing which again leads to emerging rents via falling real wages. These two processes both have in common that the real wage has to be lower (at least temporarily) to reach the full-employment situation.

Another possibility to reach full-employment seems to be to switch to more labour-intensive production techniques, which implies a rising b. But this does not do the job! With sticky real wages this measure would only raise the ratio $\frac{w}{p}b$ which would lead to a contraction of the economy in the long run. Only if the real wage rate would be adjusted for lower productivity so that $\frac{w}{p}b$ remains constant unemployment may diminish. Please note again, that this solution implies that the real wage is lower for all time. An analogous analysis can be derived for an influx of new labour (migration).

4.2 Constant growth economy

Above we have seen that - given initial conditions - the model exhibits a full-employment growth path at a growth rate

$$\gamma_q = \gamma_b + \gamma_k$$

Further the condition

$$\frac{w}{p}b = (1-a) - (\gamma_b + \gamma_k)$$

has to be satisfied. Analogously to the constant economy case we can state that the economy is growing at a lower rate if

$$(1-a) - (\gamma_b + \gamma_q) < \frac{w}{p}b$$

and the economy has the potential to grow faster if

$$(1-a) - (\gamma_b + \gamma_q) > \frac{w}{p}b$$

In the case that the economy is starting at the full-employment level this means, in the first case, that the actual growth rate is lower than the necessary growth rate to keep the economy at the full-employment level (the economy is constrained by the available intermediate inputs). In the second case the potential growth rate of the economy is higher than the necessary growth rate to keep the economy at the full-employment level. In this situation the economy is in a sense constrained by a labour (and consumption) shortage. If investors are reacting to this, less will be invested than is available as retained earnings (rents) and the difference may be consumed, exported, or saved in some form which does not lead directly to productive investment.

4.3 Emerging underemployment equilibria

4.3.1 Constant economy

We now discuss the case with $\gamma_k = \gamma_b = 0$ and with the economy at the full-employment level. What happens if there is a sudden and once for all drop in the productivity level to $b_1 < b_0$? First of all $\frac{w}{p}b$ would fall and thus $(1-a) > \frac{w}{p}b_1$. Thus there would be a potential for growth as intermediate inputs are saved. If these are actually reinvested the economy may expand and come back to full employment. Now assume that the nominal wage rate is adjusted to this increase in labour productivity such that $\frac{w_1}{p}b_1 = \frac{w_0}{p}b_0$. In this case the economy would produce exactly the same amount as before the technological shock but there are less workers employed. If these unemployed workers have no influence on wage setting they will remain unemployed forever.

4.3.2 Growing economy

4.4 Conclusions

What can be learned from this? To achieve full-employment in an economy with has to change its growth path (due to increased productivity or population growth) two conditions have to be satisfied:

1. Rents have to emerge (in our model by falling real per unit wages) to enable the economy to grow.

2. These rents have to be reinvested despite falling real wages and thus potentially lower consumption demand.

3. Higher effective demand comes from investment which widens the productive capacities.

5 Traverse analysis

5.1 Full-employment path

Above we have seen that keeping the economy at the full employment level the condition $\overline{\gamma}_q = \gamma_b + \gamma_k$ has to be satisfied. In this section we discuss the implications for the wage movements in the case that technical progress and labour growth is not constant but that the growth rates change over time. As we are focusing on transitory dynamics in the following we simplify the analysis by setting the long-run mark-up to zero, i.e. $\pi = 0$.

$$\overline{\gamma}_q(t) = \frac{1}{p(t)}m(t) = \frac{(1-a)p(t) - w(t)b(t)}{p(t)} = (1-a) - \frac{w(t)}{p(t)}b(t)$$

has to be satisfied. In the subsequent we assume that prices are fixed, i.e. $\dot{p} = 0$ to study the implications for wage setting. For fixed prices this can also be seen as real wage policy.

Thus the condition above reduces to

$$\overline{\gamma}_q(t) = (1-a) - \frac{w(t)}{p} b(t)$$

and thus

$$w(t) = \frac{1}{b(t)} p\big((1-a) - \overline{\gamma}_q(t)\big)$$

i.e. the wage rate at time t is determined by the productivity level and the growth rates $\overline{\gamma}_q(t) = \gamma_b(t) + \gamma_k(t)$. Differentiating with respect to time gives

$$\begin{aligned} \dot{w}(t) &= -p\frac{1}{b(t)^2}\Big((\dot{\gamma}_b + \dot{\gamma}_k)b(t) - \dot{b}(t)(\gamma_b + \gamma_k) + \dot{b}(t)(1-a)\Big) \\ &= -p\frac{1}{b(t)}\left((\dot{\gamma}_b + \dot{\gamma}_k) - \gamma_b^2 - \gamma_b\gamma_k + \gamma_b(1-a)\right) \end{aligned}$$

Rearranging and using the condition $\frac{w(t)}{p}b(t) = (1-a) - (\gamma_b + \gamma_k)$ yields

$$\frac{\dot{w}}{w} = -\gamma_b - (\dot{\gamma}_b + \dot{\gamma}_k)\big((1-a) - (\gamma_b + \gamma_k)\big)^{-1}$$

A similar condition holds for fixed wage rates and flexible prices

$$\frac{\dot{p}}{p} = \gamma_b + (\dot{\gamma}_b + \dot{\gamma}_k)\big((1-a) - (\gamma_b + \gamma_k)\big)^{-1}$$

If both, wage rate and price, are flexible the condition is

$$\frac{\dot{w}}{w} - \frac{\dot{p}}{p} = -\gamma_b - (\dot{\gamma}_b + \dot{\gamma}_k)\big((1-a) - (\gamma_b + \gamma_k)\big)^{-1}$$

Note, that for $\dot{\gamma}_b = \dot{\gamma}_k = 0$ these conditions are the same as stated in sections 3.4 and 3.5, respectively.

5.2 Dynamic formulation of prices and the emergence of rents

Price adjustment can be modeled as a price to cost (plus normal mark-up) adjustment

$$\dot{p}(t) = \delta_p\big[(1+\pi)\left(p(t)a(t) + w(t)b(t)\right) - p(t)\big]$$

with $0 < \delta_p \leq 1$ being the adjustment parameter. With $\delta_p < 1$ and a positive technology shock ($a(t)$ or $b(t)$ falling) rents emerge in addition to profits.

We now model the distribution of rents. As rents emerge, one can allow a certain proportion of such rents to be distributed to workers. In that case the portion of rents at the disposal of capital owners is

$$(1-\kappa_s)s(t) = (1-\kappa_s)\left[p(t) - (1+\pi)c(t)\right]$$

where $0 \leq \kappa_s \leq 1$ denotes the share of unit rents which are given to workers (see also wage equation below). Total profits then have a normal mark-up and a rent component

$$m(t) = r(t) + (1-\kappa_s)s(t)$$

to which we refer as 'retained profits'. As above, $\kappa_m m(t)q(t)$ is spent for consumption and $(1-\kappa_m)m(t)q(t)$ for investment.

5.3 Labour market dynamics

The dynamics of the wage rate is now modeled as follows:

$$\dot{w} = \kappa_s \frac{s(t)}{b(t)} + \kappa_u u(t) w(t)$$

where $u(t)$ denotes the unemployment rate

$$u(t) = \frac{k(t) - l(t)}{k(t)}$$

We assume that $0 \leq \kappa_s \leq 1$ and $\kappa_u \leq 0$. The first term means that part of transitory rents are distributed to workers (e.g. for compensating them for the increases in productivity) and the second term imposes a negative effect of unemployment on the growth of the nominal wage rate.

For reasons which will become clear below we assume that population $n(t)$ grows at a constant rate γ_n, i.e.

$$\frac{\dot{n}(t)}{n(t)} = \gamma_n$$

Then labour supply is modeled as

$$\dot{k}(t) = \delta_k \big(l(t) - k(t) \big) + \delta_n \big(k(t) - \zeta n(t) \big)$$

where ζ is the long-run participiation rate.[6] We assume that

$$\delta_k = \begin{cases} \delta_{k,IN} > 0 & \text{if } l(t) - k(t) \geq 0 \\ \delta_{k,OUT} \geq 0 & \text{if } l(t) - k(t) < 0 \end{cases}$$

The actual participation rate may differ in the short to medium run from the long-term rate ζ and is defined as

$$\frac{k(t)}{n(t)} = \frac{l(t) + u(t)}{n(t)}$$

Of course, the natural constraint of labour supply is $k(t) \leq n(t)$.

5.4 Quantity dynamics

Finally we come to the dynamics of the quantity system outside the steady-state.

With the assumption that all incomes are spent the growth path is determined as before, i.e.

$$\dot{q}(t) = (1 + \overline{\gamma}_q)(1 - a)^{-1} \big((1 - \kappa_m) f_m(t) + \kappa_m f_m(t) + f_w(t) \big) - q(t)$$

[6] The model could also allow to include a changing participation rate, i.e. $\zeta(t)$ but we shall not introduce this in this paper.

where the modification lies in the determination of spending on accumulation and the part of rents distributed to wages:

$$\overline{\gamma}_q = (1-\kappa_m)\frac{f_m(t)}{q(t)} = (1-\kappa_m)\frac{\frac{m(t)}{p(t)}q(t)}{q(t)} = (1-\kappa_m)\frac{m(t)}{p(t)} = (1-\kappa_m)\frac{r(t)+(1-\kappa_s)s(t)}{p(t)}$$

This expression can be further reduced to

$$\overline{\gamma}_q = (1-\kappa_m)\left[(1-a) - \frac{w(t)b(t)}{p(t)} - \kappa_s(1+\pi)\left(\frac{1-(1+\pi)a}{1+\pi} - \frac{w(t)b(t)}{p(t)}\right)\right]$$

In equilibrium, i.e. $\frac{wb}{p} = \frac{1-(1+\pi)a}{1+\pi}$, the economy is growing at the rate $\overline{\gamma}_q = (1-\kappa_m)\frac{\pi}{1+\pi}$. This can be verified by inserting the equilibrium real per unit wage in the equation above. In general, the emergence of rents $s(t)$ allows the economy to grow faster as more intermediate goods are available for investment allowing a faster growth rate. On the other hand the real wage will be lower (but employment rises). Note that the demand condition is satisfied in any case as it does not matter if demand is for consumption or for investment.

In disequilibrium with $s(t) > 0$ and thus the actual real wages below the equilibrium level, i.e. $\frac{w(t)b(t)}{p(t)} < \frac{1-(1+\pi)a}{1+\pi}$, the economy is able to grow faster. On the other hand, if part of the rents are (immediately) redistributed to workers as $\kappa_s \geq 0$, this again lowers the growth rate as can be seen in the equation above. For $\kappa_s = 0$ the equation reduces to the cases discussed above. For $\kappa_s = 1$, which means that all rents are distributed to the workers, the equation reduces to

$$\overline{\gamma}_q = (1-\kappa_m)\pi\frac{p(t)a + w(t)b(t)}{p(t)}$$

In equilibrium this determines to the growth rate as $\overline{\gamma}_q = (1-\kappa_m)\frac{\pi}{1+\pi}$.

5.5 The impact of technical change

In this section we explore the quantity, price, and wage dynamics through the simulation of the model. Labour productivity improvements are introduced in a S-shaped form.

TO BE INCLUDED

6 Financial assets, distributional dynamics and effective demand

In this section we continue with transitional dynamics. We shall first analyse the impact of technical change. We shall show that the impact of technical change on the growth path of an economy depends on the parameters which determine the distribution of rents, through price adjustments and adjustments in the labour market (such as changes in the participation rate). We shall then show below that the distribution of rent income between workers and capitalists has further repercussions on the growth path if the assumption is dropped that all income is being spent. We introduce what we call a 'whimp'-factor which reflects the hesitancy of capitalists to proceed with investment spending when the growth of demand is not assured. Dropping the assumption that all incomes are being spent leads us to introduce the possibility of saving in the form of the acquisition of financial assets.

6.1 Saving in financial assets

We now allow for the possibility that capitalists are not spending all their income on goods purchases. So far, we have discussed that earnings from total profits are either reinvested or consumed. Both have exerted a demand effect (either for investment goods or consumption goods). The effect was that consumption out of profits reduces the growth rate of the economy as less is reinvested. We now introduce the possibility that part of the earnings are invested in financial assets and thus do not have a direct demand effect. This represents a 'leakage' out of the real part of the economy. It also expresses the potential instability of investment demand, i.e. the hesitancy of ploughing back income into productive investment when there is no assurance of rising demand. This further contributes to the possibility that actual growth falls short of potential growth.

Total profit and interest income now amounts to

$$\big(r(t) + (1-\kappa_s)s(t)\big)q + \rho z$$

where ρ is the interest rate per unit for a financial asset and z is the stock of financial assets.

These retained earnings are distributed across different uses in the following way:

1. Spending for consumption: $\kappa_m\Big(\big(r(t) + (1-\kappa_s)s(t)\big)q + \rho z\Big)$
2. Spending for investment: $(1-\kappa_m)\Big(\big(r(t) + (1-\kappa_s)s(t)\big)q + \rho z\Big)$

   Investment is either in productive capacities or in financial assets:

   (a) Productive capacities: $(1-\eta)(1-\kappa_m)\Big(\big(r(t) + (1-\kappa_s)s(t)\big)q + \rho z\Big)$

   (b) Financial assets: $\eta(1-\kappa_m)\Big(\big(r(t) + (1-\kappa_s)s(t)\big)q + \rho z\Big)$

η denotes the share of investments in financial assets. If $\eta < 0$ this would mean that there is an 'injection' of funds from the financial to the real sector, allowing for additional investment demand.

To concentrate on the effects of the whimp factor η we assume that $\kappa_m = 0$ and $\kappa_s = 0$. Further we assume in a first step that earnings in the financial sector ρz do not flow back to the real sector (alternatively one can assume at this stage that $\rho = 0$). Demand for investment goods in the real sector reduces to

$$(1-\eta)\frac{m(t)q}{p(t)}$$

and the investment ratio which determines the growth rate is

$$(1-\eta)\frac{m(t)}{p(t)}$$

6.1.1 Growth effects of financial investments

The growth path of the economy becomes now

$$\dot{q}(t) = \left(1+(1-\eta)\frac{m}{p}\right)(1-a)^{-1}\left((1-\eta)\frac{m}{p}+\frac{w}{p}b\right)-q$$

Let us assume that $\frac{m}{p}>0$ which is fixed and constant. Real wages can be written as

$$\frac{w}{p}b = (1-a)-\frac{m}{p}$$

Inserting into the growth equation and rearranging yields

$$\begin{aligned}\frac{\dot{q}}{q} &= \left(1+(1-\eta)\frac{m}{p}\right)(1-a)^{-1}\left((1-a)-\eta\frac{m}{p}\right)-1\\ &= \frac{(1-\eta)(1-a)-\eta}{(1-a)}\frac{m}{p}-\frac{\eta(1-\eta)}{(1-a)}\left(\frac{m}{p}\right)^2\end{aligned}$$

This implies an inverse U-shaped relationship between η and retained (profit) earnings $\frac{m}{p}$ (see figure 6.1). Setting $\frac{\dot{q}}{q}=0$ gives the critical value of real retained earnings: larger

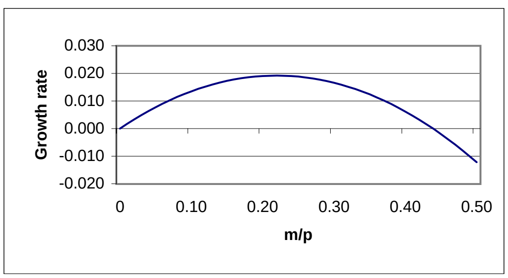

Figure 6.1: Relationship between real profits and growth rate

earnings would even mean a negative growth rate.

$$\left(\frac{m}{p}\right)^{cr} = \frac{(1-\eta)(1-a)-\eta}{\eta(1-\eta)}$$

Differentiating the growth rate with respect to $\frac{m}{p}$ and setting to zero yields the growth maximising (the second derivative is negative) real retained earnings:

$$\left(\frac{m}{p}\right)^{max} = \frac{(1-\eta)(1-a)-\eta}{2\eta(1-\eta)}$$

The reason for an inverse U-shaped relationship between $\frac{m}{p}$ and γ_q is the following: On one hand we know that, given the classical assumption that all investment spending is made out of profits, there can be no positive growth without profit income. Consequently, initially, a redistribution from wages to profits must generate growth (as long as $\eta < 1$). However at high values of η the 'leakage' out of profits reduces the level of 'effective demand'. On the other hand, wage income which in the above formulation is specified as the residual income always has a real demand effect albeit being spent only on consumption.

[TO BE CONCLUDED]

6.1.2 Simulations

[TO BE INCLUDED]

7 Conclusions

A Stability analysis

A.1 Summary of the model

The model can be summarized in the following system of differential equations:

$$\begin{aligned}
\dot{p} &= \delta_p p(t)(a-1) + \delta_p w(t) b \\
\dot{q} &= \left((1-a) - \frac{w(t)}{p(t)} b\right) q(t) \\
\dot{w} &= \kappa_u \left(1 - \frac{bq(t)}{k}\right) w(t)
\end{aligned}$$

As we have seen above the dynamics of the model depends on the behaviour of the real wage $\frac{w(t)}{p(t)}$. We first thus discuss the following simplified version of the model:

$$\begin{aligned}
\dot{v} &= \delta_v \left(\frac{1-a}{b} - v(t)\right) + \kappa_u \left(1 - \frac{b}{k} q(t)\right) \\
\dot{q} &= \big((1-a) - v(t) b\big) q(t)
\end{aligned}$$

The first equation gives the dynamic of the real wage v. Here, the first term means that the real wage converges to the equilibrium value $\frac{1-a}{b}$ whereas the second term reflects the labour market effect. In the case of unemployment $1 - \frac{b}{k} q(t) > 0$ there is a pressure on the real wage as $\kappa_u < 0$. The system is non-linear because of the term $v(t)q(t)$ in the quantity equation. Setting $\dot{v}$ and $\dot{q}$ equal to zero yields the fix-point

$$\begin{aligned}
v^* &= \frac{1-a}{b} \\
q^* &= \frac{k}{b}
\end{aligned}$$

Linearizing the system and evaluating at the fix-point gives the Jacobi matrix $\mathbf{J}^*$.

$$\mathbf{J}^* = \begin{pmatrix} -\delta_p & -\kappa_u \frac{1-a}{k} \\ -k & 0 \end{pmatrix}$$

The eigenvalues of the system can be calculated by

$$\lambda_{1,2} = \frac{1}{2} \left(\text{tr } \mathbf{J}^* \pm \sqrt{(\text{tr } \mathbf{J}^*)^2 - 4 \det \mathbf{J}^*} \right)$$

where $\text{tr } \mathbf{J}^* = -\delta_p \leq 0$ denotes the trace of the matrix and $\det \mathbf{J}^* = -\kappa_u (1-a) \geq 0$ is the determinant.

	$\boldsymbol{\delta > 0}$	$\boldsymbol{\delta = 0}$	$\boldsymbol{\delta < 0}$
$\boldsymbol{\kappa > 0}$	$tr < 0$ $det < 0$ $\Delta > 0$ $\lambda_1 > 0$ $\lambda_2 < 0$	$tr = 0$ $det < 0$ $\Delta > 0$ $\lambda_1 > 0$ $\lambda_2 < 0$	$tr > 0$ $det < 0$ $\Delta > 0$ $\lambda_1 > 0$ $\lambda_2 > 0$
$\boldsymbol{\kappa = 0}$	$tr < 0$ $det = 0$ $\Delta > 0$ $\lambda_1 = 0$ $\lambda_2 < 0$	$tr = 0$ $det = 0$ $\Delta = 0$ $\lambda_1 = 0$ $\lambda_2 = 0$	$tr > 0$ $det = 0$ $\Delta > 0$ $\lambda_1 > 0$ $\lambda_2 = 0$
$\boldsymbol{\kappa < 0}$	$tr < 0$ $det > 0$ (1) $\Delta > 0$ $\lambda_1 < 0$ $\lambda_2 < 0$ (2) $\Delta = 0$ $\lambda_1 < 0 =$ $\lambda_2 < 0$ (3) $\Delta < 0$ $Re\lambda_1 < 0 =$ $Re\lambda_2 < 0$	$tr = 0$ $det > 0$ $\Delta < 0$ $Re\lambda_1 = 0 =$ $Re\lambda_2 = 0$	$tr > 0$ $det > 0$ (1) $\Delta > 0$ $\lambda_1 > 0$ $\lambda_2 > 0$ (2) $\Delta = 0$ $\lambda_1 > 0 =$ $\lambda_2 > 0$ (3) $\Delta < 0$ $Re\lambda_1 > 0 =$ $Re\lambda_2 > 0$

Table A.1: Stability analysis

B The multisectoral model

In this section we shortly discuss the main properties of the model in the multisectoral case in the steady-state (balanced growth). For an application of the multi-sectoral model with internationally integrated economies see Stehrer (2001) which is based on an earlier version of the model introduced in Landesmann and Stehrer (2000). Here we assume that the technology (input-output matrix) is given and the labour input coefficients are also fixed. Further we assume that wage rates for each type of worker z w_i^z are set exogenously and constant. This is the case if the following conditions are satisfied: First, there are no transitory rents, $\mathbf{s}' = \mathbf{0}$ (what will be the case when prices are in equilibrium), or the parameters $\kappa_{s_i} = 0$ for all i, which means that transitory rents does not have an effect on wage rates. Second, wages are equalised across sectors, $w_i^z = w_j^z$ for all i, j, or $\kappa_{w^z} = 0$, which means that there is no wage equalisation across sectors. Third, unemployment has no effect on wages, either because $\kappa_{u^z} = 0$ or labour supply is perfectly elastic. We first discuss only the model for a closed economy and then show how the results can be applied to integrated economies.

B.1 Technology

Technology is given and denoted by an input-output matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & \dots & a_{N2} \\ \vdots & \ddots & \vdots \\ a_{1N} & \dots & a_{NN} \end{pmatrix}$$

and labour productivity is given by a vector

$$\mathbf{a}_l^{z'} = (a_{l1}^z, \dots, a_{lN}^z)$$

where z denotes the skill-type of worker.

B.2 Prices

The price system is modeled as a simple system of differential equations where prices adjust to unit costs.

$$\dot{\mathbf{p}}' = (1 + \pi)\left(\mathbf{p}'\mathbf{A} + \boldsymbol{\omega}'\right) - \mathbf{p}' \tag{B.1}$$

where $\mathbf{p}'$ is a vector of prices, $\mathbf{A}$ is the technology matrix, and $\boldsymbol{\omega}' = (w_i a_{li}, \dots, w_N a_{lN})$ is a vector of unit labour costs for each industry. We assume that wages w_i and labour input coefficients a_{li} can be different across sectors. If there are more skill-types of workers the vector of unit labour costs is

$$\boldsymbol{\omega}' = \left(\sum_z w_i^z a_{li}^z, \dots, \sum_z w_N^z a_{lN}^z\right)$$

or with the assumption that skill-specific wages are equalised across sectors

$$\boldsymbol{\omega}' = \left(\sum_z w^z a_{li}^z, \ldots, \sum_s w^s a_{lN}^s \right)$$

π gives the long-run mark-up rate, which is assumed to be equal across sectors.[7] Setting $\dot{\mathbf{p}}' = \mathbf{0}$ gives the equilibrium price vector

$$\mathbf{p}' = (1+\pi)\left(\mathbf{p}'\mathbf{A} + \boldsymbol{\omega}'\right)$$

which can be solved for

$$\mathbf{p}^{*'} = (1+\pi)\boldsymbol{\omega}'\left[\mathbf{I} - (1+\pi)\mathbf{A}\right]^{-1} \tag{B.2}$$

It should be noted that solving the system of differential equations above directly yields the same result for exogenously given wages and the technology parameters.[8]

B.3 Profits and rents

The per unit profits in each sector are defined as a mark-up on costs

$$\mathbf{r}' = \pi \mathbf{c}'$$

Further in disequilibrium there are rents which are defined as:

$$\mathbf{s}' = \mathbf{p}' - (1+\pi)\mathbf{c}'$$

In equilibrium these rents are zero as one can show by inserting the equilibrium price vector:

$$\begin{aligned}
\mathbf{s}' &= \mathbf{p}^{*'} - (1+\pi)\left(\mathbf{p}^{*'}\mathbf{A} + \boldsymbol{\omega}'\right) \\
&= (1+\pi)\boldsymbol{\omega}'\left[\mathbf{I} - (1+\pi)\mathbf{A}\right]^{-1} - (1+\pi)(1+\pi)\boldsymbol{\omega}'\left[\mathbf{I} - (1+\pi)\mathbf{A}\right]^{-1}\mathbf{A} - \\
&\quad (1+\pi)\boldsymbol{\omega}'\left[\mathbf{I} - (1+\pi)\mathbf{A}\right]\left[\mathbf{I} - (1+\pi)\mathbf{A}\right]^{-1} \\
&= (1+\pi)\boldsymbol{\omega}'\left[\mathbf{I} - (1+\pi)\mathbf{A}\right]^{-1}\Big(\mathbf{I} - (1+\pi)\mathbf{A} - \big(\mathbf{I} - (1+\pi)\mathbf{A}\big)\Big) \\
&= \mathbf{0}'
\end{aligned}$$

In the following we define a vector $\mathbf{R}$ which adds up profits and rents

$$\mathbf{R}' = (\mathbf{r}' + \mathbf{s}') = \mathbf{p}' - \mathbf{c}'$$

In the case that these transitory rents are not zero we have to assume that $\kappa_{s_i} = 0$ for all i if $s_i > 0$ for guaranteeing constancy of wages as mentioned above. By this assumption we can also show the more general case where profits and rents need not to be equalised across sectors, either because of different s_i or long-term sector-specific mark-ups π_i.

[7] This assumption is not necessary from a technical point of view, although it is quite common in the literature where the equalisation of profits across sectors is assumed. Differences in the profitability of sectors in the model discussed in this paper may come from differences in transitory rents s_i.

[8] For given wages and labour input coefficients this is a non-homogenous system of differential equations with a constant coefficient matrix, in general $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{c}$. The solution is given by $\mathbf{x}^* = -\mathbf{A}^{-1}\mathbf{c}$. Further the system is stable if the eigenvalues of the matrix $(1+\pi)\left[\mathbf{p}'\left(\mathbf{A} - \mathbf{I}\right)\right]$ are negative. Given the assumptions on the technology matrix this is a stable system.

B.4 The quantity system

Next we discuss the quantity system. Here we have to assume that $\dot{\mathbf{p}} = \mathbf{0}$. Thus, the results presented assume stable prices, although these need not be equilibrium prices.

Demand consists of three different components: First there is demand for intermediate goods used in production, $\mathbf{Aq}$, where $\mathbf{q}$ denotes the vector of quantities. Second there is a matrix of demand from profits

$$\mathbf{D}_R\mathbf{q} = \begin{pmatrix} \beta_1 \frac{R_1}{p_1} & \dots & \beta_1 \frac{R_N}{p_1} \\ \vdots & \ddots & \vdots \\ \beta_N \frac{R_1}{p_N} & \dots & \beta_N \frac{R_N}{p_N} \end{pmatrix} \mathbf{q}$$

β_i denotes the nominal share of profit expenditure in sector i with $\sum_i \beta_i = 1$, and R_i the per unit profit plus rent in each sector. A typical element of the vector $\mathbf{D}_R\mathbf{q}$ is $\beta_i \frac{\sum_j R_j}{p_i}$. This specification would also hold, if only part of the sum of profits and rents are spent. Thus investment expenditures out of profits depend on nominal expenditure shares β_i and (relative) prices. One has to note here, that the nominal shares β_i only describe the outcome of investment behaviour at the aggregate level.

The third source of demand comes from wage income. Consumption expenditures out of wages are denoted in matrix form

$$\mathbf{D}_W\mathbf{q} = \begin{pmatrix} \alpha_1 \frac{\sum_z w_1^z a_{l1}^z}{p_1} & \dots & \alpha_1 \frac{\sum_z w_N^z a_{lN}^z}{p_1} \\ \vdots & \ddots & \vdots \\ \alpha_N \frac{\sum_z w_1^z a_{l1}^z}{p_N} & \dots & \alpha_N \frac{\sum_z w_N^z a_{lN}^z}{p_N} \end{pmatrix} \mathbf{q}$$

α_i are the nominal shares in consumption with $\sum_i \alpha_i = 1$. The specific assumption in this formulation is that workers are maximising a Cobb-Douglas utility function, which is linear-homogenous and homothetic. This means that all workers have the same (constant) nominal shares of consumption. A more more general specification of the demand (e.g. dependent on real income levels and prices) could be used here. For given wage rates and prices the nominal shares $\alpha_i^z(w_i^z, \mathbf{p})$ would then also be constant although differing across skill types of workers and, in the case of wage differentiation across sectors, differ across skill types and sectors. A typical element in the matrix would then be $\sum_z \alpha_{i,j}^z \frac{w_i^z a_{li}^z}{p_j}$ where $\sum_j \alpha_{i,j}^z = 1$. We do not explore this general case here. Total demand is the sum of these three components

$$\begin{aligned} \mathbf{q}^D &= \mathbf{Aq} + \mathbf{D}_R\mathbf{q} + \mathbf{D}_W\mathbf{q} \\ &= (\mathbf{A} + \mathbf{D}_R + \mathbf{D}_W)\,\mathbf{q} \end{aligned} \tag{B.3}$$

In the following we set $(\mathbf{A} + \mathbf{D}_R + \mathbf{D}_W) = \mathbf{\Omega}$. In equilibrium we must have $\mathbf{q}^D = \mathbf{q}$ and thus the expression above must satisfy

$$(\mathbf{\Omega} - \mathbf{I})\,\mathbf{q} = \mathbf{0}$$

This is a linear-homogenous system which has a non-trivial solution, $\mathbf{q} \neq \mathbf{0}$, if

$$\det(\boldsymbol{\Omega} - \mathbf{I}) = 0$$

The determinant of a matrix equals zero if the columns or rows are linearly dependent.[9] In this case at least one row (or column) is a linear combination of the other rows (columns). The linear dependency can be shown by multiplying a particular column k in the matrix above with the price vector $\mathbf{p}'$ which gives

$$\sum_j p_j a_{ji} - p_k + \sum_j \beta_j R_k + \sum_j \alpha_j \omega_k = \sum_j p_j a_{ji} - p_k + R_k + \omega_k$$

where we used the condition that $\sum_i \alpha_i = \sum_i \beta_i = 1$. In equilibrium total income must equal total expenditure, thus $\mathbf{R}' + \boldsymbol{\omega}' = \mathbf{A} - \mathbf{p}'\mathbf{A}$ or $R_k + \omega_k = p_k - \sum_j p_j a_{ji}$. Inserting this into the equation gives

$$\sum_j p_j a_{ji} - p_k + R_k + \omega_k = \sum_j p_j a_{ji} - p_k + p_k - \sum_j p_j a_{ji} = 0$$

This shows the linear dependency of at least one row on the others and therefore there exists a non-trivial solution, i.e. an ouput vector $\mathbf{q} \neq \mathbf{0}$. Here, two important features should be mentioned: First, this condition is generally true as long as $\sum_i \alpha_i = \sum_i \beta_i = 1$. Thus the condition does not depend on a particular formulation of investment or consumption demand. The only necessary condition is, that the nominal shares resulting from the underlying decision structure sum to unity. Second, a solution exits also at non-equilibrium price vectors where investments out of rents then come from both profits and rents, $r_i + s_i$, respectively. Solving the system of equations determines the structure of the output but not the level of economic activity.

Further one may show that there exist a non-negative solution to the problem. This can be done in two steps. First, we sum up the rows of the matrix $\boldsymbol{\Omega}$ and, second, show that the column sums are equal to one, thus

$$\boldsymbol{\iota}'\boldsymbol{\Omega} = \boldsymbol{\iota}'\mathbf{I}$$

Premultiplying $\boldsymbol{\Omega}$ and $\mathbf{I}$ with a matrix $\boldsymbol{P}$ which contains the prices p_i in the diagonal yields

$$\begin{aligned} \boldsymbol{\iota}'\boldsymbol{P}\boldsymbol{\Omega} &= \boldsymbol{\iota}'\boldsymbol{P}\mathbf{I} \\ \mathbf{p}'\boldsymbol{\Omega} &= \mathbf{p}'\mathbf{I} \end{aligned}$$

This can be rewritten as

$$\mathbf{p}'(\mathbf{A} - \mathbf{I} + \mathbf{D}_R + \mathbf{D}_W) = \mathbf{0}'$$

Rearranging gives

$$\mathbf{p}'\mathbf{D}_R - \mathbf{p}'[\mathbf{I} - (\mathbf{A} + \mathbf{D}_W)] = \mathbf{0}'$$

[9] Please note, that this condition is analogue to the condition of the existence of a solution in the closed Leontief model.

Using

$$\mathbf{p}'\mathbf{D}_R = \left(\sum_i \beta_i R_1, \ldots, \sum_i \beta_i R_N\right) = \mathbf{R}'$$

and

$$\mathbf{p}'\mathbf{D}_W = \left(\sum_i \alpha_i w_1 a_{l1}, \ldots, \sum_i \alpha_i w_N a_{lN}\right) = \boldsymbol{\omega}'$$

and inserting gives

$$\mathbf{R}' - [\mathbf{p}' - (\mathbf{p}'\mathbf{A} + \boldsymbol{\omega}')] = \mathbf{R}' - \mathbf{R}' = \mathbf{0}'$$

which again shows the existence of a non-trivial solution. Accordingly to the Perron-Frobenius theorems the maximum eigenvalue of $\mathbf{\Omega}$ is $\lambda_{\mathbf{\Omega}}^{max} = 1$ of which the components of the associated eigenvector are non-negative.

The dynamics of the supply of goods is modeled as a system of supply-adjusts-to-demand differential equations

$$\dot{\mathbf{q}} \;=\; (1+g)\,[\mathbf{I}-\mathbf{A}]^{-1}\,(\mathbf{D}_R + \mathbf{D}_W)\,\mathbf{q} - \mathbf{q} \qquad \text{(B.4)}$$

Inserting for $(\mathbf{D}_R + \mathbf{D}_W) = (\mathbf{I} - \mathbf{A})$, which is satisfied in equilibrium, gives

$$\dot{\mathbf{q}} = (1+g)\,[\mathbf{I}-\mathbf{A}]^{-1}\,[\mathbf{I}-\mathbf{A}]\,\mathbf{q} - \mathbf{q} = g\mathbf{q}$$

Thus the quantity system grows at a constant rate g (steady-state balanced growth path).

We have to analyse the relationship between the demand (and supply) for investment goods $\mathbf{D}_R\mathbf{q}$ and the growth rate g. The system is constant (in the case $R_i = 0$, or $\pi_i = 0$ and $s_i = 0$ for all $i = 1, \ldots, N$) or is growing at a constant rate g where

$$g = \min_i \left(\frac{q_i^I}{q_i}\right)$$

and

$$q_i^I = \beta_i \sum_j \frac{R_j q_j}{p_i}$$

Here we assume the non-negativity of $\mathbf{R}$. As we have stated in the main text, the optimal structure of investment is given for

$$\beta_i^* = \frac{p_i q_i}{\mathbf{p}'\mathbf{q}}$$

To show this we insert the quantities demanded for investment into the definition of the growth rate

$$g = \min_i \left(\beta_i \frac{\mathbf{R}'\mathbf{q}}{p_i q_i}\right)$$

where $\sum_i \beta_i = 1$. As we want to maximise the growth rate this is rewritten

$$g^* = \max_{\beta_i} \left[\min\left(\beta_i \frac{\mathbf{R}'\mathbf{q}}{p_i q_i}\right)\right]$$

This problem has the solution $\beta_i^* = \frac{p_i q_i}{\mathbf{p}'\mathbf{q}}$ for given $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{R}$ and $\sum_i \beta_i = 1$. We show this by using the condition that $\beta_i \frac{\mathbf{R}'\mathbf{q}}{p_i q_i} = \beta_j \frac{\mathbf{R}'\mathbf{q}}{p_j q_j}$ for all i, j. If this condition is satisfied for all but two sectors, e.g. $\beta_i \frac{\mathbf{R}'\mathbf{q}}{p_i q_i} > \beta_j \frac{\mathbf{R}'\mathbf{q}}{p_j q_j}$ then g would be constrained by sector j. In this case the growth rate can be increased by lowering β_i and raising β_j. Using this condition and the normalisation $\sum_i \beta_i = 1$ the problem can be solved easily: Multiplying the terms $\beta_i \frac{\mathbf{R}'\mathbf{q}}{p_i q_i}$ by $\frac{\mathbf{p}'\mathbf{q}}{\mathbf{R}'\mathbf{q}}$ (normalisation) and writing them as a system of equations

$$\boldsymbol{\Theta}\boldsymbol{\beta} = \boldsymbol{\iota}$$

where $\boldsymbol{\Theta}$ is a matrix with the terms $\frac{\mathbf{p}'\mathbf{q}}{p_i q_i}$ in the diagonal and $\boldsymbol{\iota}$ is a vector of ones implementing the condition that $\beta_i \frac{\mathbf{R}\mathbf{q}}{p_i q_i} = \beta_j \frac{\mathbf{R}\mathbf{q}}{p_j q_j}$. Solving this system of equations yields

$$\boldsymbol{\beta}^* = \boldsymbol{\Theta}^{-1}\boldsymbol{\iota} = (\mathbf{p}'\mathbf{q})^{-1} (p_1 q_1, \ldots, p_N q_N)'$$

Thus the structure of nominal shares must be equal to the structure of output. If this condition is not satisfied, then there would be excess investment in all but the sector with the lowest $\frac{q_i^I}{q_i}$ and the growth rate is bounded by this sector. Inserting β_i^* in the formulation of the growth rate yields

$$g^* = \frac{\mathbf{R}'\mathbf{q}}{\mathbf{p}'\mathbf{q}}$$

In equilibrium (i.e. with $\mathbf{s} = \mathbf{0}$ or at prices $\mathbf{p}^*$) this can be reformulated as

$$g^* = \frac{\mathbf{r}'\mathbf{q}}{\mathbf{p}'\mathbf{q}} = \frac{\pi\mathbf{c}'\mathbf{q}}{(1+\pi)\mathbf{c}'\mathbf{q}} = \frac{\pi}{1+\pi}$$

If this condition is satisfied then the economy is growing in equilibrium exactly with $g = \frac{\pi}{1+\pi}$ which denotes the (equalised) profit rate in each sector.[10]

B.5 Labour demand

Labour demand is then modeled simply by

$$L^{D_z} = \mathbf{a}_l^{z'} \mathbf{q}$$

for each skill group z.

[10]This is not the maximum (von Neumann) growth path g^{max}. For this case consumption would have to be at zero level.

References

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