Too little or too much R&D?

Maria Jose Alvarez* and Christian Groth**

* Universidad de Málaga, Universidad de Alicante; alvarezp@uma.es

** Institute of Economics, University of Copenhagen; Chr.Groth@econ.ku.dk

Corresponding author:

Christian Groth, Institute of Economics, University of Copenhagen,

Studiestraede 6, DK-1455, Copenhagen K, Denmark

tel: (+ 45) 35 32 30 28, fax: (+ 45) 35 32 30 00

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Abstract

According to the first generation models of endogenous growth based on expanding product variety, the market economy unambiguously yield a too low level of R&D. However, disentangling returns to specialization from the market power parameter, Benassy found that this result arises due to the implicit choice of a relatively high value for the returns to specialization. The present paper takes a step further, analyzing growth based on expanding product variety in a framework where returns to specialization, the markup, and the capital share in final output are given by independent parameters. We show, first, that low returns to specialization are not a necessary condition for excessive R&D in the market economy; rather, the decisive factor is the implicit presence of negative disaggregate externalities of increased specialization. Second, even if R&D is socially useless, the market equilibrium may involve allocation of ressources to R&D. Empirically, an advantage of the more general framework is better agreement with the observed level of markups and the observed falling tendency of the patent/R&D ratio.

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1 Introduction

One of the central questions in innovation-based growth theory is whether a decentralized market economy tends to undertake too little or too much R&D compared with the socially optimal level. Two strands of models exist, seemingly giving different answers to this question. An unambiguously too low level of research was implied by the first generation of *expanding variety* (or horizontal innovation) models such as Romer (1990) and Grossman and Helpman (1991a), surveyed in Barro & Sala-i-Martin (1995, ch. 6). As an alternative line, the *quality ladder* models of vertical innovation (e.g., Aghion and Howitt, 1992) added "creative destruction" or "business stealing" to the analysis, thereby concluding that whether a too low or too high level of R&D arises, depends on whether "social surplus creation" and "intertemporal knowledge spillovers" dominate the "creative destruction" effect.

Benassy (1998) argues that the apparent asymmetry between the two strands of models is due to an implicit choice of a particular value for the "returns-to-specialization parameter". Indeed, in the first generation expanding variety models, the returns to specialization as well as the monopolistic markup charged by the innovative sector are captured by the same parameter. Extending the Grossman-Helpman version of the expanding variety model without capital accumulation (Grossman and Helpman 1991a), Benassy finds that once the parameter for the returns to specialization is chosen independently of the monopolistic markup[1], a too *high* growth rate can appear if (and only if) the returns to specialization is sufficiently smaller than implied by the "Romer-style" models.

In the present paper we draw attention to the relevance of an additional separation of parameters. We study an expanding variety model with capital accumulation where the returns to specialization, the markup, and the capital share in manufacturing are given by independent parameters so that the two above-mentioned aspects are not only disentangled from each other but also from the capital share in manufacturing. Previous expanding variety models (where returns to variety occurs within the production sphere) can be seen as special cases. An implication of the general framework is that the separation of the share of capital from the markup is crucial for the comparative statics as well as the conditions under which an inefficiently high level of R&D can appear. Indeed, low returns to specialization are not necessary

[1]As noted by Benassy this parameter separation is in fact present in the original contribution on endogenous product variety by Ethier (1982).

for excessive R&D in the market economy. Whatever the returns to specialization, the decisive factor is the implicit presence of enough "creative down-weighing", a reminiscence of "creative destruction".

This is shown by placing the horizontal innovation models, with the natural parameter separations included, on a similar footing as the vertical innovation models, that is, by decomposing the repercussions of innovations into different types of "effects": an "intertemporal spillover", a "direct surplus effect", and an externality on other producers. This externality on other producers is of ambiguous sign, however; it may take the form of "creative down-weighing" (a mild form of "creative destruction") or the form of "creative synergy", depending on circumstances. Interestingly, when "creative down-weighing" completely offsets the direct surplus effect and the intertemporal spillover (so that R&D is socially useless), the market equilibrium may nevertheless have resources allocated to R&D.

Empirically, the separation of the markup from the share of capital (or more generally the share of intermediate goods) allows the markup to take values more in line with the evidence, thereby avoiding a weakness of the Romer-style expanding variety models. In these models where the monopolistic markup is directly linked to the share of intermediate goods in manufacturing output, unrealistically high values of the markup tend to arise. In addition, with independence of the returns-to-specialization parameter better agreement with the observed falling tendency of the patent/R&D ratio can be obtained.

On one point the present model has similarity with the chapter-5 version of the increasing variety model in Grossman and Helpman (1991b). That model also implies a separation of the monopolistic markup from the capital share in manufacturing. But the expanding product variety feature is limited to non-durable intermediate goods, and returns to specialization are implicitly given as a function of the share of these intermediate goods and the markup parameter. This (together with further simplifications) leads to the reappearance of the Romer result that there is always too little R&D generated in the decentralized economy.

While our purpose is to trace out analytical aspects of first generation models rather than transcending them, the paper by Jones and Williams (2000) is definitely a second generation paper. Recognizing that "creative destruction" can be a byproduct of horizontal innovations as well as of vertical innovations, Jones and Williams include full-fledged creative destruction in a horizontal innovation model (in addition to duplication externalities along the lines of Jones (1995)). Just as a better quality of a given good can replace the old quality, a new good may functionally replace existing goods and outsell these. The setup of the Jones and Williams paper is one of *semi-endogenous* growth, and though the model loosens the restrictive link between returns to specialization and the markup as well as the link between the markup and the capital share, there is no complete disentangling of these parameters in their model. The focus is not on the analytical relationships opened up by parameter separations, but on calibrating the model for the US economy. The conclusion is that the decentralized economy vastly underinvests in R&D relative to what is socially optimal. The Stokey (1995) paper, which is in the "quality ladder" tradition, is less firm about this matter.

The remainder of our paper is organized as follows. Section 2 introduces the elements of an increasing variety model with two sectors along the lines of the first generation models. There are a basic-goods sector and an innovative sector, called the specialized capital-goods sector. R&D takes place within the specialized capital-goods sector. Section 3 considers the control problem of the social planner and solves for the optimal steady state. Section 4 embeds the economic system into a market economy. In addition, this section shows how different contributions to the literature can be seen as special cases of the model. In Section 5 we compare the balanced growth properties of the market economy with those of the social optimum. Section 6 relates the model to the empirics of markups and the trend of the patent/R&D ratio. Finally, Section 7 concludes.

2 Elements of the economy

2.1 Preferences and technology

The economy is populated by a constant number, L, of infinitely lived identical households with constant size. Each household supplies one unit of labor inelastically. Their preferences can be represented by a discounted utility function,

$$\int_0^\infty e^{-\rho t}\frac{c(t)^{1-\theta}-1}{1-\theta}dt, \quad \rho>0, \theta>0,$$

where $c(t)$ is consumption at time t, θ is the elasticity of marginal utility, and ρ is the rate of time preference [2].

[2] In case $\theta = 1$, the expression $\frac{c^{1-\theta}-1}{1-\theta}$ should be interpreted as $\ln c$.

The economy has two production sectors: the basic-goods sector and the specialized capital-goods sector. In the *basic-goods sector,* labor and specialized capital goods are the inputs to produce the aggregate output, $Y(t)$, according to a CRS production function. In accordance with the literature referred to in the introduction we limit the analysis to the Cobb-Douglas case:

$$Y(t) = A(t)^{\eta} X(t)^{\alpha} N_Y(t)^{1-\alpha}, \qquad 0 < \alpha < 1, \quad \eta > 0, \tag{2.1}$$

where $A(t)$ is a measure of the level of technical knowledge in society or the stock of engineering principles that grows through research. Each "unit of knowledge" corresponds to a design of a specialized capital good. The designs and the corresponding specialized capital goods are numbered chronologically, $i = 1, 2, ..., A(t)$. The parameter η captures the elasticity of output with respect to knowledge, sometimes called "the returns to specialization", i.e., the degree to which society benefits from specializing production in an increasing number of branches. Each new branch corresponds to a new design, that is a new specialized capital good.

$N_Y(t)$ is the labor input in the basic-goods sector at time t, and $X(t)$ is a CES aggregate of quantities of the existing specialized capital goods:

$$X(t) = A(t)\Big(\frac{1}{A(t)} \sum_{i=1}^{A(t)} x_i(t)^{\varepsilon}\Big)^{\frac{1}{\varepsilon}}, \qquad 0 < \varepsilon < 1. \tag{2.2}$$

Thus the existing specialized capital goods exhibit a constant (direct) elasticity of substitution given by $\frac{1}{1-\varepsilon} > 1$ implying that no specialized capital good is essential. A higher ε indicates larger substitutability between the specialized capital goods; hence, we call ε the substitution parameter. Moreover, in a symmetric state where all the specialized capital goods are used in the same amount, i.e., $x_i(t) = x(t)$ for all i, the aggregate $X(t)$ of specialized capital goods is a linear function of $A(t)$, $X(t) = A(t)x(t)$. Notice that (2.2) inserted into (2.1) shows the specialized capital goods to be complements in the production of basic goods ($\frac{\partial^2 Y}{\partial x_i \partial x_j} > 0$) if $\alpha > \varepsilon$ and to be supplements ($\frac{\partial^2 Y}{\partial x_i \partial x_j} < 0$) if $\alpha < \varepsilon$.

The output of basic goods is used for consumption, $C(t) \equiv c(t)L$, and investment in "raw" capital, $I(t)$, i.e., $Y(t) = C(t)+I(t)$. The stock of raw capital $K(t)$ increases according to $\dot{K}(t) = I(t) - \delta K(t)$, where $\delta > 0$ is the capital depreciation rate[3]. Hence,

$$\dot{K}(t) = Y(t) - C(t) - \delta K(t), \qquad K(0) = K_0, \tag{2.3}$$

[3] A dot over a variable signifies the derivative with respect to time t.

where K_0 is a given positive number.

In the *specialized capital-goods sector,* which is also the *innovative sector,* a unit of raw capital can immediately be transformed to a specialized capital good on the basis of a given technical design. The number of new designs created per time unit is assumed proportional to research work. And research efficiency is assumed proportional to the existing stock of knowledge, measured by $A(t)$ (the standing-on-shoulders effect, also present in Romer 1990 and Benassy 1998). Hence, ignoring indivisibility,

$$\dot{A}(t) = \gamma N_A(t)A(t), \qquad A(0) = A_0, \tag{2.4}$$

where A_0 is a given positive number, $\gamma > 0$ is a productivity parameter, and $N_A(t)$ is aggregate research work[4]. Finally, with full employment,

$$N_Y(t) + N_A(t) = L. \tag{2.5}$$

In what follows, to simplify notation we suppress the explicit dating of the variables when not needed for clarity. Because of the strict concavity of X in x_i and the symmetric cost structure, static efficiency requires $x_i = x$ for $i = 1, 2, ...A.$[5] From this follows, first, $X = Ax$ from (2.2). Second, the stock demand for raw capital can be written Ax and since the supply is K, when demand equals supply we have

$$X = Ax = K. \tag{2.6}$$

Henceforth we assume static efficiency. Inserting (2.6) into (2.1) gives output of basic goods as

$$Y = A^{\eta} K^{\alpha} {N_Y}^{1-\alpha}. \tag{2.7}$$

To summarize, the restrictions on the parameters are (unless otherwise indicated)

$$0 < \alpha < 1, \quad 0 < \varepsilon < 1, \quad \theta, \rho, \eta, \delta, \gamma, L > 0. \tag{P}$$

[4] A slightly more robust specification would allow "diminishing diminishing returns" with respect to A, along the lines of Jones and Manuelli (1990) and Dalgaard and Kreiner (2001). That kind of specification implies $\partial^2 \dot{A}/\partial A^2 < 0$, but in such a way that $lim_{A \to \infty} \partial \dot{A}/\partial A > 0$. As far as steady state analysis is concerned, this would not change much, apart from a substitution of asymptotic steady states for ordinary steady states. Hence, we shall stick to (2.4).

Another simplification in (2.4) is the omission of duplication externalities, cf. Jones (1995). Though possibly quantitatively important in practice, this kind of externalities does not seem inherent in the logic of expanding variety models.

[5] Therefore, obsolescence of old capital goods never occurs.

2.2 Direct and indirect effects of specialization

By introducing a specific parameter η to capture the returns to specialization and by letting the substitution parameter ε be independent of the capital elasticity of output, α, we have obtained the desired disentangling of parameters that are dependent in Romer-style models. These models (Romer 1990, Barro and Sala-i-Martin 1995, ch. 6.1.7) have, implicitly, $\eta = 1 - \alpha$ and $\varepsilon = \alpha$, this last condition implying a kind of intermediate case where the specialized capital goods are neither complements nor substitutes[6]. Interestingly the two conditions together imply that the focus is on the benchmark case of "zero indirect effects" of new inventions.

To make clear what is meant by this, we introduce a conceptual distinction between A, the number of existing different varieties at time t, and A_0, the number of these varieties being in use at time t (that is, the number of varieties for which $x_i > 0$). Consider the situation just before the invention of a new variety. As mentioned, static efficiency requires $A_0 = A$ and $x_i = \bar{K}/A_0 \equiv x$, where $\bar{K}$ is a given aggregate amount of raw capital. We may write (2.1) as

$$Y = A^{\eta+(1-\frac{1}{\varepsilon})\alpha}(\sum_{i=1}^{A_0} x_i^{\varepsilon})^{\frac{\alpha}{\varepsilon}} N_Y^{1-\alpha} = A^{\eta+(1-\frac{1}{\varepsilon})\alpha} A_0^{\frac{\alpha}{\varepsilon}} x^{\alpha} N_Y^{1-\alpha} \equiv F(A, A_0, x, N_Y). \qquad (2.8)$$

Now, after the invention of the new variety, static efficiency requires a *redistribution* of the given amount of raw capital to an enlarged spectrum of varieties. The effect on aggregate output of this redistribution is called the *direct effect* of increased specialization. This direct effect is $F_{A_0} + F_x \frac{\partial x}{\partial A_0} = (\frac{\alpha}{\varepsilon} - \alpha)\frac{Y}{A}$. In view of the assumption $0 < \varepsilon < 1$, this effect is always positive.

There may also be an *indirect effect* of increased specialization, that is, an effect on the productivity of the already existing specialized capital goods. To calculate this effect, we freeze A_0 and x at the level they have before redistribution. Then we define the indirect effect as

$$F_A = \left[\eta + (1 - \frac{1}{\varepsilon})\alpha\right] \frac{Y}{A} \equiv \xi \frac{Y}{A}.$$

One can imagine this effect to be positive, zero, or negative depending on the circumstances. We shall speak of "creative synergy" when it is positive ($\xi > 0$), and of "creative down-weighing" (a mild form of the "creative destruction" inherent in vertical innovation models) when it is negative ($\xi < 0$). The interpretation of the

[6]It should be recognized that Romer (1990, p. S81) actually encouraged an investigation of cases where $\varepsilon \neq \alpha$.

first case is that the direct contribution of the invention is complemented by the positive indirect effect due to the other capital goods becoming more productive when "assisted" by a more complete network of intermediate goods. The interpretation of the second case is that though the overall effect of an invention on capital productivity is positive (since $\eta > 0$, by assumption), the direct contribution of the invention is partly offset by a negative indirect effect due to, say, coordination difficulties in a more specialized and complex world.

Expressing these direct and indirect productivity effects as elasticities, the overall effect of inventions can be written

$$\frac{A}{Y}\frac{\partial Y}{\partial A} = \text{direct effect} + \text{indirect effect} = (\frac{1}{\varepsilon} - 1)\alpha + \xi = \eta. \tag{2.9}$$

Hence, we may interprete returns to specialization η as a *derived* parameter, given α, ε, and the indirect effect $\xi > -(\frac{1}{\varepsilon} - 1)\alpha$. Depending on the sign of ξ, we shall speak of η as *total* or *net* returns to specialization.

Romer's parameter links imply the benchmark case where the indirect effect vanishes (since $\xi = \eta - (\frac{1}{\varepsilon} - 1)\alpha = 0$ when $\eta = 1 - \alpha$ and $\varepsilon = \alpha$).

Sometimes it will be convenient to let technical progress be represented in the "labor-augmenting" way and write output of basic goods as

$$Y = K^{\alpha}(A^{\nu}N_Y)^{1-\alpha}, \tag{2.10}$$

where $A^{\nu}N_Y \equiv E$ is the "effective" labor input. Then, $\nu \equiv \frac{\eta}{1-\alpha}$ is the elasticity of labor efficiency with respect to technical knowledge, i.e., a co-determinant of the rate of labor-augmenting technical progress. This invites a decomposition analogous to (2.9),

$$\frac{A}{E}\frac{\partial E}{\partial A} = \frac{1}{1-\alpha}\frac{A}{Y}\frac{\partial Y}{\partial A} = \text{direct effect} + \text{indirect effect} = \frac{1-\varepsilon}{1-\alpha}\frac{\alpha}{\varepsilon} + \frac{\xi}{1-\alpha} = \nu. \tag{2.11}$$

When $\nu = \frac{1-\varepsilon}{1-\alpha}\frac{\alpha}{\varepsilon}$, "zero indirect effects" of inventions obtains. The Romer case, $\eta = 1 - \alpha$ and $\varepsilon = \alpha$, implying $\nu = 1 = \frac{1-\varepsilon}{1-\alpha}\frac{\alpha}{\varepsilon}$, is a subcase of this.

2.3 The family of efficient steady states

In this paper the focus will be on steady states. Some definitions:

Definition 1 *Given* $K_0 > 0$ *and* $A_0 > 0$, *a path* $(K, A, C, Y, N_Y, N_A)_{t=0}^{\infty}$ *is called feasible if: (a)* K *and* A *are continuous functions of* t; *(b)* C, Y, N_Y, *and* N_A *are piecewise continuous functions of* t; *(c) the path satisfies (2.7) and (2.5) for all* $t \geq 0$, *and it satisfies (2.3) and (2.4) for all* $t \geq 0$ *except at points of discontinuity of* C, N_Y, *and* N_A; *and* K, A, C, Y, N_Y, *and* N_A *are non-negative for all* $t \geq 0$.

Definition 2 *A feasible path* $(K, A, C, Y, N_Y, N_A)_{t=0}^{\infty}$ *is called a steady state if* K, A, C, *and* Y *are strictly positive and grow at constant though not necessarily equal (or positive) rates.*

Let the rate of growth of a strictly positive variable x be denoted g_x, i.e., $g_x \equiv \dot{x}/x$. Let u be the fraction of total labor supply employed in the basic-goods sector, i.e., $N_Y \equiv uL$, $0 \leq u \leq 1$. By (2.4), $g_A \geq 0$ always.

Lemma 1 *(i) In a steady state with* $g_A = \bar{g}_A$, $u = 1 - \frac{\bar{g}_A}{\gamma L}$, *a constant. Moreover,* $0 < u \leq 1$ *and* $0 \leq \bar{g}_A < \gamma L$. *(ii) If, in addition,* Y/K *is constant*[7], *then* $g_c = g_Y = g_K \equiv \bar{g} = \frac{\eta}{1-\alpha}\bar{g}_A$.

Proof. See Appendix 8.1. ■

Observe that, from (2.7), $\partial Y/\partial K = \alpha Y/K$ is constant when Y/K is constant. We shall concentrate on dynamically efficient steady states.

Definition 3 *A steady state with* $g_C = \bar{g}$ *and* Y/K *constant is called dynamically efficient if* $\partial Y/\partial K - \delta \geq \bar{g}$.

To prepare the ground for a comparison of steady states in a market economy with those in a social optimum we make the following observation. Given the initial level of technical knowledge A_0, a dynamically efficient steady state is completely characterized by the consumption growth rate $\bar{g}$ and the output-capital ratio Y/K. To be more precise:

Proposition 1 *For any parameter constellation* $(\alpha, \varepsilon, \eta, \theta, \rho, \delta, \gamma, L)$ *satisfying* (P) *and any initial knowledge level* $A_0 > 0$ *there exists a 2-dimensional family of dynamically efficient steady states with constant* Y/K *through time. This family is*

[7] A steady state commonly has constant Y/K. But there is an exception. If $I = 0$, i.e., $C = Y$ and $g_K = -\delta$, then, by (2.7), $g_C = g_Y = \eta g_A + \alpha g_K = \eta g_A - \alpha\delta > -\delta$ so that Y/K increases.

indexed by the consumption growth rate $\bar{g} \in \left[0, \frac{\eta}{1-\alpha}\gamma L\right)$ *, and the output-capital ratio* $Y/K \geq (\bar{g}+\delta)/\alpha$ *(or, equivalently, the investment-output ratio* $I/Y \in (0, \alpha]$*). Moreover, all these steady states are distinct in the sense that either the growth rates or the consumption levels at any* $t \geq 0$ *(or both) are different across them. In addition, given* $\bar{g}$*, the consumption level at any* $t \geq 0$ *is a decreasing function of* Y/K*. Finally, given* $\bar{g}$ *and* Y/K*, larger initial knowledge* A_0 *implies larger consumption level at any* $t \geq 0$*.*

Proof. See Appendix 8.1. ■

3 The social optimum

The social planner will of course ensure static efficiency. Therefore, in the social optimum, manufacturing output is given by (2.7). The social planner chooses a path $(c, N_Y)_{t=0}^{\infty}$ to

$$\max_{c\geq 0, 0\leq N_Y \leq L} \int_0^\infty e^{-\rho t} \frac{c^{1-\theta}-1}{1-\theta}\, dt \qquad \text{s.t.} \tag{3.1}$$
$$\dot{K} = A^\eta K^\alpha {N_Y}^{1-\alpha} - \delta K - cL, \quad K(0) = K_0 > 0 \text{ given,}$$
$$\dot{A} = \gamma(L - N_Y)A, \quad A(0) = A_0 > 0 \text{ given,}$$

and subject to the non-negativity requirements: $A, K \geq 0$ for all $t \geq 0$.

From the first order conditions for an interior solution to the problem (3.1) we find (see appendix 8.1) the marginal product of capital in a steady state to be[8]

$$\frac{\partial Y}{\partial K}^* = \alpha \frac{Y}{K} = \frac{\eta}{1-\alpha}\gamma L + \delta, \tag{3.2}$$

and the rate of growth of consumption to be

$$g_c^* = \frac{1}{\theta}\left(\frac{\eta}{1-\alpha}\gamma L - \rho\right). \tag{3.3}$$

Hence, the optimum rate of growth does not depend on the substitution parameter ε; this parameter gets a role only through the market forces considered in the next section. However, an increase in the (net) returns to specialization parameter, η, raises the degree to which society benefits form new inventions, which leads to an increase in g_c^*. In addition, the inverse of the labor share in manufacturing output,

[8]Variables describing the social planner's solution are marked by *.

$1-\alpha$, acts as a "multiplier" transforming returns to specialization into an elasticity of labor efficiency with respect to technical knowledge. Or, considering the explicit labor-augmenting version, (2.10), of the production function where, $\nu \equiv \frac{1}{1-\alpha}\eta$ is the elasticity of labor efficiency with respect to technical knowledge. Then, given η, this elasticity decreases when the labor share in output, $1-\alpha$, increases, thereby lowering the growth rate in a steady state. Moreover, the higher is the desire for smoothing consumption (higher θ) and the higher is the rate of impatience (larger ρ), the smaller is g_c^*. In addition, the growth rate increases with the size of population; this is the well-known, though controversial, "scale effect" of R&D-based endogenous growth models[9].

Definition 4 *A steady state with positive employment in both sectors is called an interior steady state.*

In the derivation of the expression for g_c^* above, an interior steady state is presupposed. That is, the inequalities $0 < g_c^* < \frac{\eta}{1-\alpha}\gamma L$ are presupposed (the second inequality arising from (ii) of Lemma 1 combined with (2.4)). Using (3.3), we see that $g_c^* < \frac{\eta}{1-\alpha}\gamma L$ holds if and only if

$$\rho > (1-\theta)\frac{\eta}{1-\alpha}\gamma L, \tag{A1}$$

and $g_c^* > 0$ holds if and only if

$$\rho < \frac{\eta}{1-\alpha}\gamma L \tag{A2}$$

The conclusion is that, given A1 and A2, there exists a unique steady state[10], and it satisfies $0 < g_c^* < \frac{\eta}{1-\alpha}\gamma L = \frac{\partial Y}{\partial K}^* - \delta$ (obviously it is dynamically efficient). The steady state is saddlepoint stable, and the unique converging path *is* indeed an optimal solution (Appendix 8.1). If, however, A1 is violated, the rate of time preference is so small that the system cannot avoid the temptation to specialize in

[9] We have to assume that the population is constant (in the long run at least) in order to avoid explosive growth to arise in this class of models. An alternative route would have been to shift the attention to *semi-endogenous* growth (as in, e.g., Jones 1995). But in this paper we stick to a given class of *endogenous* growth models in order to trace out some unnoticed aspects of this framework.

[10] We define uniqueness of a steady state to be present if, given the parameters $(\alpha, \varepsilon, \theta, \rho, \eta, \delta, \gamma, L)$, there exists only one pair $(g_c, Y/K)$ which is consistent with the steady state conditions.

R&D activity forever (thus postponing production of consumption goods forever); in this case no optimal solution exists. On the other hand, if A2 is violated, then impatience is so large that there will be no R&D activity and no growth in a steady state. The formulas (3.2) and (3.3) are no longer valid; instead, the steady state solution of the model will be like that of a standard one-sector Ramsey model without technical progress[11].

Now, we will embed the economic system in a market economy. Apart from the more general specifications of technology, the set-up is similar to Romer (1990).

4 The market economy

4.1 Firms

In the market economy, the supply of *basic goods* is decided by profit maximizing firms operating under perfect competition facing the technology (2.1)-(2.2). The representative firm in the sector rents labor at the wage w and specialized capital goods at the rental rate R_i, $i = 1, 2, ..., A$ (as above we suppress the explicit dating of the variables unless needed for clarity). Using basic goods as our *numeraire,* profit maximization yields

$$\frac{\partial Y}{\partial N_Y} = (1-\alpha)\frac{Y}{N_Y} = w, \tag{4.1}$$

$$\frac{\partial Y}{\partial x_i} = \alpha \frac{Y}{X}\frac{\partial X}{\partial x_i} = R_{i.} \tag{4.2}$$

From (4.2) we can express the demand for the specialized capital good i conditional on a given X as:

$$x_i = \frac{X}{A}(\frac{R_i}{R})^{-\frac{1}{1-\varepsilon}}, \qquad i = 1, 2, ..., A, \tag{4.3}$$

where $R = (\frac{1}{A}\sum_{i=1}^{A} R_i^{\frac{\varepsilon}{\varepsilon-1}})^{\frac{\varepsilon-1}{\varepsilon}}$ is the "ideal" price index for X (the minimum cost per unit of X).

The supply of each specialized capital good is decided by the firm that invented the design for the capital good in question, i.e., firm i supplies capital good variety i. The firms get compensated for the sunk research cost through retention of monopoly power over the commercial use of the invention and this monopoly power is supported

[11] As is well-known, also this steady state is unique.

by patents of infinite duration. Hence, the capital-goods sector operates under monopolistic competition[12]. Since, given design i, to deliver $x(i)$ units of capital good i, it takes $x(i)$ units of raw capital, the cost per unit of capital good i is $r+\delta$, where r is the real interest rate of the market. At each instant of time, firm i, facing the demand curve (4.3) and taking X and R as given, sets the rental rate R_i so that current profit $\pi_i \equiv R_i x_i - (r+\delta)x_i$ is maximized, given the interest rate r. The resulting monopoly rental price is a simple markup over the marginal cost, the markup being determined by the elasticity of demand,

$$R_i = \frac{1}{\varepsilon}(r+\delta). \tag{4.4}$$

Smaller ε (indicating less substitutability between specialized capital goods) gives larger monopoly power to the suppliers of specialized capital goods.

Due to the specification of the production function (2.1) and the way specialized capital goods enter, the markup, $\frac{1}{\varepsilon}$, and the share of capital goods in output, α, are independent. This separation is in contrast to Romer-style models, but partly in line with the model of chapter 5 in Grossman & Helpman (1991b). That model ignores, however, the other important separation, that between returns to specialization and the markup parameter. Further, the model has only one kind of capital, the expanding product variety feature being limited to non-durable intermediate goods. In addition, labor is the only input needed in the sector producing these intermediate goods. One can show that these traits lead to the reappearance of the Romer result that there is always too little R&D generated in the decentralized economy. The purpose of the present paper is to expose and explain, in a more general setup, the conditions under which this is not the case.

Since, by (4.4), all firms in the specialized capital-goods sector set the same rental price $R \equiv \frac{1}{\varepsilon}(r+\delta)$, they supply the same quantity, x, and they earn the same profit

$$\pi = \frac{1}{\varepsilon}(r+\delta)x - (r+\delta)x = (\frac{1}{\varepsilon} - 1)(r+\delta)x. \tag{4.5}$$

In addition the static efficiency condition (2.6) holds.

The equilibrium value of a patent is given by the present discounted value of the revenue that the patent generates

$$p(t) = \int_t^\infty (\frac{1}{\varepsilon} - 1)(r(\tau)+\delta)x(\tau)e^{-\int_t^\tau r(s)ds}d\tau. \tag{4.6}$$

[12] At initial time, $t=0$, the number of firms, $A(0)$, is large enough so that each firm's action is negligible in the aggregate economy.

Differentiating this expression with respect to t (using Leibniz' rule) yields the no-arbitrage condition

$$\frac{\pi + \dot{p}}{p} = r, \tag{4.7}$$

i.e., the return on a patent must be equal to the return on capital.

There is free entry to research activity. Research is done by new firms wanting to enter the specialized capital-goods sector. Given the invention production function (2.4), the value of the marginal product of labor in research is $p\gamma A$. Hence, profit maximization subject to (2.4) entails, in equilibrium,

$$w \geqq p\gamma A, \qquad \text{with} = \text{if } N_A > 0. \tag{4.8}$$

Once a new technical design has been invented, a patent is taken out and the new firm starts supplying the corresponding new specialized capital good. Although the right to supply a specific capital good is patented, everybody has access to the scientific principle behind the new design, i.e., technical knowledge is considered only partially excludable. In other words, by increasing A, research activity has a positive external effect on the productivity of future research activity[13]. In addition, research activity has a positive overall effect on total factor productivity in manufacturing (through the term A^{η} in (2.7)), this effect being the sum of the direct effect and the (possibly negative) indirect effect of increased specialization as described in Section 2.2.

4.2 Households

There are L infinitely lived households, each supplying one unit of labor inelastically. The households consume and save, and savings can be either in capital or in shares of the monopoly firms. Financial wealth of the representative household is $v \equiv \frac{K+pA}{L}$. The household makes a plan $(c)_{t=0}^{\infty}$ to solve

$$\max_{c \geq 0} \int_0^{\infty} e^{-\rho t} \frac{c^{1-\theta} - 1}{1-\theta} dt \qquad \text{s.t.} \tag{4.9}$$

$$\dot{v} = w + rv - c, \quad v(0) = v_0, \text{ given,} \tag{4.10}$$

$$\lim_{t \to \infty} v e^{-\int_0^t r ds} \geq 0, \tag{4.11}$$

[13] Each research firm is small and therefore perceives, correctly, its contribution to aggregate $\dot{A}$ to be practically negligible.

where the last constraint is the no-Ponzi-game condition. Necessary and sufficient conditions for a solution are that the Keynes-Ramsey rule,

$$g_c = \frac{1}{\theta}(r - \rho), \tag{4.12}$$

and the transversality condition,

$$\lim_{t\to\infty} ve^{-\int_0^t rds} = 0, \tag{4.13}$$

hold for all $t \geq 0$.

4.3 General equilibrium

Clearing conditions in the capital and labor markets entail: $K = X = xA$, $L = N_A + N_Y$. We define the *effective* capital-labor ratio in the basic-goods sector as $\tilde{k} \equiv K/(A^{\frac{\eta}{1-\alpha}}uL)$, where u is the fraction of total labor supply employed in the basic-goods sector. Then

$$x = \frac{K}{A} = uL\tilde{k}A^{\frac{\eta+\alpha-1}{1-\alpha}}. \tag{4.14}$$

Output per unit of *effective* labor in the basic-goods sector is $\tilde{y} \equiv Y/(A^{\frac{\eta}{1-\alpha}}uL)$ $= \tilde{k}^{\alpha}$. With these definitions and the clearing conditions together with (4.1), (4.2), and (4.4), we have

$$w = (1-\alpha)\frac{Y}{uL} = (1-\alpha)\tilde{k}^{\alpha}A^{\frac{\eta}{1-\alpha}}, \tag{4.15}$$

$$\frac{1}{\varepsilon}(r+\delta) = \frac{\partial Y}{\partial K} = \alpha\frac{Y}{K} = \alpha\tilde{k}^{\alpha-1}. \tag{4.16}$$

If $u < 1$, i.e., $N_A > 0$, then w also equals the value of the marginal product of labor in research so that (4.8) reduces to $w = p\gamma A$. This together with (4.15) gives the market value of a patent as

$$p = \frac{1-\alpha}{\gamma}\tilde{k}^{\alpha}A^{\frac{\eta+\alpha-1}{1-\alpha}}. \tag{4.17}$$

To summarize, an *equilibrium* for this market economy is a path for prices and quantities such that (i) consumers maximize discounted utility, i.e., they satisfy the Keynes-Ramsey rule and the transversality condition, taking the path of interest rates and wage rates as given; (ii) basic-goods producers maximize profits choosing inputs of labor and a list of specialized capital goods taking the prices of these inputs as given; (iii) each firm in the specialized capital-goods sector chooses a rental price to maximize profits, taking as given the interest rate and the downward-sloping demand curve it faces, (iv) R&D activity takes place only when (4.8) holds with equality; (v) in each market, the supply is equal to the demand. An *interior equilibrium* is an equilibrium such that $0 < u < 1$ (there is positive employment in both sectors).

4.4 Steady state

In an interior steady state the growth rate of consumption is

$$g_c = \frac{(1-\varepsilon)\alpha\gamma L - (1-\alpha)\rho}{1-\varepsilon\alpha-(1-\theta)\eta}\frac{\eta}{(1-\alpha)}, \tag{4.18}$$

and the marginal product of capital is

$$\frac{\partial Y}{\partial K} = \alpha\frac{Y}{K} = \frac{1}{\varepsilon}(\theta g_c + \rho + \delta). \tag{4.19}$$

These formulas are derived in Appendix 8.2 and presuppose two parameter restrictions. First, the transversality condition (4.13) will be satisfied if and only if

$$\rho > \frac{1-\theta}{1-\varepsilon\alpha}\eta\frac{1-\varepsilon}{1-\alpha}\alpha\gamma L. \tag{A3}$$

If this does not hold, then the rate of time preference is so small that the market economy tends to grow at a rate above the interest rate, and household wealth tends to infinity, thus violating the equilibrium assumption.

An equally important parameter restriction comes from the interiority condition $u < 1$ (or $g_A > 0$) which holds if and only if

$$\rho < \frac{1-\varepsilon}{1-\alpha}\alpha\gamma L. \tag{A4}$$

If, however, A4 is violated, then impatience is so large that R&D activity and growth cannot be supported in a steady state equilibrium. In this case the steady state solution of the model is like that of a one-sector model without technical progress. Also this zero-growth steady state is unique (since $\frac{Y}{K} = \frac{1}{\alpha\varepsilon}(r+\delta) = \frac{\rho+\delta}{\alpha\varepsilon}$ from (4.16) and (4.12)).

Observe that the parameter restriction A4, not containing either η or θ, is of a quite different nature compared to A2 of the previous section[14]. On the other hand, if A1 is assumed, we don't have to worry about A3. Indeed:

[14] The implications of this are laid open in Section 5, in particular Proposition 6.

Lemma 2 *Given (P), then: (i) A1 implies A3; (ii) A3 and A4 imply that* $1 - \varepsilon\alpha > (1-\theta)\eta$*; (iii) A1 and A3 are satisfied automatically when* $\theta \geq 1$*.*

Proof. See Appendix 8.2 ■

The implication of (ii) of the lemma is that, together, A3 and A4 rule out the possibility of a non-positive denominator of g in (4.18).

The conclusion is that, given A3 and A4, there exists a unique steady state, and it satifies $0 < g_c < \frac{(1-\varepsilon)\alpha}{1-\varepsilon\alpha}\gamma L$; further, it is dynamically efficient (since (4.16) implies $\frac{\partial Y}{\partial K} - \delta = \frac{r+(1-\varepsilon)\delta}{\varepsilon} > \frac{r}{\varepsilon} > r > g_c$, by A3). The steady state can be shown to be saddlepoint stable, at least within the empirically relevant domain of the parameter space (see Appendix 8.2).

From expression (4.18) follows that returns to specialization, η, the rate of impatience, ρ, the desire for consumption smoothing, θ, and the size of population, L, affect the growth rate qualitatively as in the social optimum (the derivations are in appendix 8.2). New features are, however, *first*, that the capital share parameter α affects growth through an additional channel compared with the social optimum. Indeed, the higher is α, the lower is the wage share in manufacturing, $1-\alpha$, which, given the factor prices, implies less room for profitable employment in the manufacturing sector[15]. Thereby more of the fixed labor force is available for employment in research, and growth is enhanced. The *second* new feature is that the growth rate of the market economy depends on the substitution parameter ε (which the social optimum did not). When specialized capital-goods are close substitutes (ε high), the markup over marginal cost in the specialized capital-goods sector becomes low, making inventions of new designs less profitable, thereby reducing growth. These two new features come from effects on private incentives.

The last-mentioned effect is blurred in Romer-style models (Romer 1990; Barro and Sala-i-Martin, 1995, ch. 6) where ε is implicitly set equal to the capital share α so that the markup is $1/\alpha$ and cannot be low unless the capital share is high. Indeed, in this special case, (4.18) reduces to $g_c = \frac{\alpha\gamma L-\rho}{1-\alpha^2-(1-\theta)\eta}\eta$ so that $\frac{\partial g_c}{\partial(1/\alpha)} < 0$, given (A2) and (A4).[16]

[15] (4.4), (4.15), and (4.16) give $uL = \frac{1-\alpha}{\alpha}K\frac{R}{w}$.

[16] Also, in Jones & Williams (2000) the link between the markup and the capital share is not entirely eliminated, since the elasticity of demand for specialized capital goods depends on the share of capital goods in final output. In addition, returns to specialization depends on both a substitution parameter and the capital share.

To sum up, there are three potential causes of market distortions present in the model: (i) The *intertemporal spillover*: By adding to the stock of technical knowledge, research increases the productivity of future research, but this effect is not compensated in the market. (ii) The *surplus appropriability problem*: Innovators capture only a fraction of their (direct) contribution to output, that is,

$$\pi = \frac{1-\varepsilon}{\varepsilon}(r+\delta)\frac{K}{A} = (1-\varepsilon)\alpha\frac{Y}{A} < \frac{1}{\varepsilon}(1-\varepsilon)\alpha\frac{Y}{A} = \frac{1}{\varepsilon}(1-\varepsilon)\frac{\partial Y}{\partial K}\frac{K}{A}, \qquad (4.20)$$

from (4.5), (4.14), (4.16), and (2.9)[17]. (iii) The *externality on other capital suppliers*: Innovators may affect the productivity of already existing capital goods positively ("creative synergy") or negatively ("creative down-weighing")[18]. The causes (i) and (ii) tend to generate insufficient research under laissez faire, while cause (iii) works in the same or the opposite direction, depending on the sign of the externality. Thus, there may be opposing effects and one cannot *a priori* "sign" the net effect on growth of the market institution relative to the social optimum.

4.5 Earlier contributions as special cases

It is convenient to let technical progress be represented in the "labor-augmenting" way. This implies writing output of basic goods as in (2.10) where the parameter $\nu \equiv \frac{\eta}{1-\alpha}$ is the elasticity of labor efficiency with respect to technical knowledge. The growth rate in an interior steady state can then be written as $g_c = \frac{(1-\varepsilon)\alpha\gamma L-(1-\alpha)\rho}{1-\varepsilon\alpha-(1-\theta)(1-\alpha)\nu}\nu$ in the market economy and as $g_c^* = \frac{1}{\theta}(\nu\gamma L - \rho)$ in the social optimum. We will henceforth consider returns to specialization, $\eta \equiv \nu(1-\alpha)$, as a derived parameter, once ν is chosen. The advantage of this is twofold. First, it makes comparison with earlier contributions to the literature easier. Second, when comparing g_c and g_c^*, given ν, none of the parameters, α and ε, enter the determination of growth in the social optimum, but only that of g_c in the market economy.

As already mentioned, the Romer (1990) model is simply the case $\nu \equiv \frac{\eta}{1-\alpha} = 1$,

[17] The surplus appropriability problem reflects that, from (4.16), capital costs are $r + \delta = \varepsilon\partial Y/\partial K < \partial Y/\partial K$. This inequality is a result of *monopoly pricing*: The markup implies a wedge between the price of the services of specialized capital goods and the marginal cost of providing them so that the demand for capital services is reduced. This also entails a wedge between social returns to saving, $\frac{\partial Y}{\partial K} - \delta$, and the private return, r, and therefore a tendency to too little saving.

[18] Note that "creative down-weighing" refers to a reduction in the *productivity* of old capital goods. This need not imply reduced profits. Indeed, along a steady state path, profits per firm are reduced by the arrival of new inventions if and only if $\nu < 1$, cf. (4.5) and (4.14).

and $\varepsilon = \alpha$[19]. The condition $\nu = 1$ has the implication that the value p of a patent and the size x of the market for a specific capital good stay constant in a steady state. To compare, if $\nu \gtrless 1$, respectively, then p and x would increase/decrease, respectively, cf. (4.17) and (4.14). The case $\nu = 1$ is also a benchmark case in the sense that $\nu = 1$ implies that along a steady state path every new invention leaves profits π of the single monopoly firm unchanged; if, however, $\nu \gtrless 1$, respectively, then π increases/decreases for every new invention as shown by (4.5) and (4.14).

The Benassy (1998) analysis, based on an extension of Grossman and Helpman (1991a), does not fit directly into our framework since the Benassy model ignores physical capital. The only input in final output production is a bundle of non-durable intermediate goods; and in the manufacturing of intermediate goods labor is the only input. Nevertheless, as far as the reduced form (4.18) is concerned, Benassy's results correspond to the case $\varepsilon = \alpha$ and $\theta = 1$ (only logarithmic utility is considered). Indeed, the technology specification of the Benassy model leads to $g_c = [(1-\omega)\gamma L - \omega\rho]\,\nu$ in the market economy. Here, the notation is the same as hitherto, apart from the new symbol ω which, in the Benassy model, is a substitution parameter such that the monopolist markup becomes $\frac{1}{\omega}$. With $\alpha = \varepsilon$ and $\theta = 1$, (4.18) gives $g_c = (\frac{\alpha}{1+\alpha}\gamma L - \frac{1}{1+\alpha}\rho)\nu$, which is the Benassy formula when we put $\omega = \frac{1}{1+\alpha}$. As to the social optimum Benassy has $g_c^* = \nu\gamma L - \rho$ which is the same as our result in Section 3 when $\theta = 1$. Extended to some general $\theta > 0$, the Benassy formula becomes $g_c = \frac{(1-\omega)\gamma L - \omega\rho}{1-\omega(1-\theta)\nu}\nu$ for the market economy (from (4.18) with $\alpha = \varepsilon$) and $g_c^* = \frac{1}{\theta}(\nu\gamma L - \rho)$ for the social optimum.

The just mentioned Grossman and Helpman (1991a) model is a special case of the Benassy model, namely the case $\nu = (1-\omega)/\omega$. As noted earlier, the book by Grossman and Helpman contains also another version of the increasing variety model, that is one with capital accumulation. This version can be interpreted as yet a special case of (the reduced form of) the Benassy model. The formula $g_c = [(1-\omega)\gamma L - \omega\rho]\,\nu$ is still valid, but ω depends not only on the markup, but also on the share of capital in final output and the share of non-durable intermediate goods in final output[20]. Also, ν is a function of the markup and these two share parameters.

[19] Strictly speaking, this refers to the textbook version (Aghion and Howitt, 1998, p. 35 ff.). In the original version of the Romer model, the basic-goods sector also employs a fixed amount of a second type of labor, but this is of secondary importance in our context.

[20] In this model physical capital is a homogenous good while the specialized intermediate goods, sold under conditions of monopolistic competition, are non-durable.

The next section studies the necessary and sufficient conditions for the market economy to do too much R&D. We shall see how allowing for $\varepsilon \neq \alpha$ modifies these conditions compared with Romer's and Benassy's results. Since the Benassy model can be seen as containing the two Grossman and Helpman contributions as special cases, our comparison with Benassy (1998) will cover these contributions as well.

5 Comparing market outcome and social optimum

Taking the knowledge elasticity ν as given and (net) returns to specialization, $\eta \equiv \nu(1-\alpha)$, as a derived parameter, we may summarize the analysis of Sections 3 and 4 in the following way:

Proposition 2 *Let the knowledge elasticity $\nu > 0$ be given.*

(i) If A1 holds, there exits a unique steady state in the social optimum and it has

$$g_c^* = \begin{cases} \frac{1}{\theta}(\nu\gamma L - \rho) > 0, & \frac{\partial g_c^*}{\partial \nu} = \frac{\gamma L}{\theta} > 0, \quad \text{if } \rho < \nu\gamma L, \\ 0 & \text{if } \rho \geq \nu\gamma L, \end{cases}$$

$$(\frac{Y}{K})^* = \frac{\theta g_c^* + \rho + \delta}{\alpha}.$$

(ii) If A3 holds, there exists a unique steady state in the market economy, it is dynamically efficient, and it has

$$g_c = \begin{cases} \frac{(1-\varepsilon)\alpha\gamma L - (1-\alpha)\rho}{1-\varepsilon\alpha-(1-\theta)(1-\alpha)\nu}\nu > 0, & \frac{\partial g_c}{\partial \nu} > 0, \frac{\partial g_c}{\partial \alpha} > 0, \frac{\partial g_c}{\partial \varepsilon} < 0, \\ & \text{if } \rho < \frac{1-\varepsilon}{1-\alpha}\alpha\gamma L, \\ 0 & \text{if } \rho \geq \frac{1-\varepsilon}{1-\alpha}\alpha\gamma L, \end{cases}$$

$$\frac{Y}{K} = \frac{\theta g_c + \rho + \delta}{\varepsilon\alpha},$$

Proof. See text of Sections 3 and 4 and the appendices 8.1 and 8.2. ■

It follows immediately from this proposition that even in cases where $g_c = g_c^*$ (hence also $N_A = N_A^*$, cf. Lemma 1), the market economy does not replicate the social optimum. Indeed, the market economy underinvests in capital, implying a too low K/Y, that is, a too high Y/K. This is because monopoly pricing ($\frac{1}{\varepsilon} > 1$) results in a low equilibrium rate of interest relative to the marginal product of capital (cf. (4.16)), and this induces too little saving and too little capital accumulation, implying in the long run too little consumption as well (cf. Proposition 1). Actually, whatever the growth rate in the market economy, the wedge between the private and the social return to saving leads to underinvestment in capital *relative* to this growth rate.

Now, let us see how resource allocation to R&D, N_A^* and N_A, in the two regimes vary as the knowledge elasticity ν increases. It is convenient to compare N_A^* with N_A (rather than g_c^* with g_c) because, like g_A, $N_A(= g_A/\gamma)$ depends on the knowledge elasticity ν in a less complex way than does g_c.

Proposition 3 *Assume A1, A2, A3, and A4. Then:*

(i) In the social optimum resource allocation N_A^* to R&D is increasing in the knowledge elasticity ν.

(ii) In the market economy resource allocation N_A to R&D is increasing in the knowledge elasticity ν if the elasticity of marginal utility $\theta < 1$; N_A is independent of ν if $\theta = 1$, and N_A is decreasing in ν if $\theta > 1$.

Proof. See Appendix 8.3. ■

The proposition is illustrated in Fig. 1. The three panels of the figure show for the cases $\theta < 1$, $\theta = 1$, and $\theta > 1$, respectively, the graphs of N_A^* and N_A, as ν varies while the other parameters are kept constant[21]. Assumption A4 is presupposed so that the numerator of g in Proposition 2 is positive for all $\nu > 0$. As to the upper panel in Fig. 1, when $\theta < 1$, then ν may be so large that A3 and/or A1 are violated so that one or both of the two steady states don't exist[22]. The explanation is that when the intertemporal elasticity of substitution, $1/\theta$, is high, and the growth potential of the economy is large relative to the rate of time preference, then the economy can run into the paradox of postponing consumption for ever.

Before discussing the intuition behind the "perverse" relationship in the lower panel of Fig. 1, we shall translate the contents of Fig. 1 into a statement comparing the *consumption* growth rates. The term *equal-growth knowledge elasticity* shall

[21] One may interprete the variations in ν as reflecting variations in the "indirect effect", ξ, of inventions, cf. (2.11).

[22] Given A4 and $\theta < 1$, the critical value, $\frac{1-\varepsilon\alpha}{(1-\theta)(1-\alpha)}$, of ν such that the denominator of g in Proposition 2 becomes zero is even larger than the value of ν for which A3 is violated, cf. Lemma 2.

denote a value $\tilde{\nu}$ such that for $\nu = \tilde{\nu}$: (a) both the social optimum and the market economy have a steady state; and (b) $g_c = g_c^*$. In view of (4.20) and the fact that $\partial Y/\partial A = \nu(1-\alpha)Y/A$, profits per capital good can be written

$$\pi = \frac{(1-\varepsilon)\alpha}{(1-\alpha)\nu}\partial Y/\partial A. \tag{5.1}$$

Therefore, a natural reference point for ν is the number $\phi(\varepsilon, \alpha) \equiv (1-\varepsilon)\alpha/(1-\alpha)$, henceforth called the "profit factor"[23].

Proposition 4 *Assume A4. Then:*

(i) There is a unique equal-growth knowledge elasticity $\tilde{\nu} > 0$.

(ii) The equal-growth knowledge elasticity $\tilde{\nu}$ *satisfies the inequality* $\frac{\rho}{\gamma L} < \tilde{\nu} < \phi(\varepsilon, \alpha)$*; further, if* $\theta < 1$*, then* $\tilde{\nu} < \frac{\rho}{(1-\theta)\gamma L}$*, while if* $\theta \geq 1$*, then* $\tilde{\nu}$ *(like* $\phi(\varepsilon, \alpha)$*) is unbounded above if* α *is free to approach 1.*

(iii) For any ν *satisfying A1, both the social optimum and the market economy have a steady state, and the following rule applies:* $g_c \gtreqless g_c^*$ *for* $\nu \lesseqgtr \tilde{\nu}$*, respectively.*

Proof. See Appendix 8.3. ■

Corollary 1 *Assume A1 and A4. The case* $g_c > g_c^*$ *arises if and only if the externality of inventions on the productivity of old capital goods is negative and numerically above* $\frac{1}{\varepsilon}\phi(\varepsilon, \alpha) - \tilde{\nu}$.

Proof. From (2.11) we have that $\nu - \frac{1}{\varepsilon}\phi(\varepsilon, \alpha)$ measures the externality on the productivity of old capital goods. By (iii) of the proposition, $g_c > g_c^*$ if and only if $\nu < \tilde{\nu}$. Hence, $g_c > g_c^*$ if and only if $\nu - \frac{1}{\varepsilon}\phi(\varepsilon, \alpha) < \tilde{\nu} - \frac{1}{\varepsilon}\phi(\varepsilon, \alpha)$, where, by (ii) of the Proposition, $\tilde{\nu} - \frac{1}{\varepsilon}\phi(\varepsilon, \alpha) < \tilde{\nu} - \phi(\varepsilon, \alpha) < 0$, since $0 < \varepsilon < 1$. ■

It so happens that unless there is enough "creative down-weighing", the market institution leads to underinvestment in R&D. To avoid this, it is not enough that creative down-weighing, ξ, is numerically above $(1/\varepsilon)(1-\varepsilon)(1-\alpha)\nu$ so that, by (2.11) and (5.1), the profit factor $\phi(\varepsilon, \alpha) \geq \nu$, hence $\pi \geq \partial Y/\partial A$ (i.e., net surplus *is* appropriated). Indeed, this only ensures that the surplus appropriability problem is overcome, not that the intertemporal spillover is overcome.

[23] Note that the upper panel in Fig. 1 has $\phi(\varepsilon, \alpha) < \frac{\rho}{(1-\theta)\gamma L}$; but the opposite inequality might just as well be true and would imply no change of the qualitative features of the graphs.

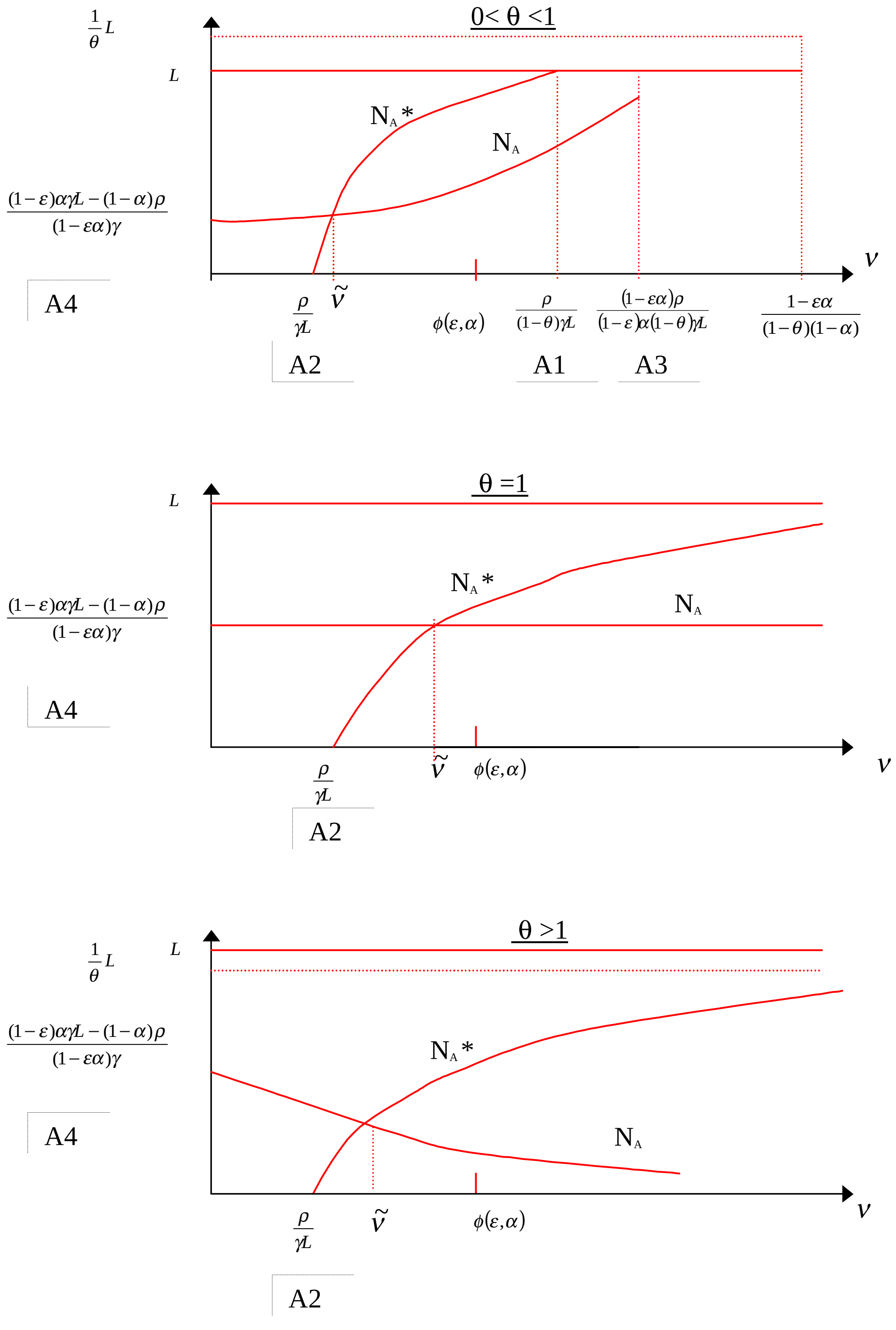

Figure 1: N_A^* and N_A as functions of the knowledge elasticity ν.

As noted in the previous section, the Romer (1990) model is the case: $\nu = 1$ and $\varepsilon = \alpha$; and the Benassy (1998) model can be interpreted as the case: ν free (though positive) and $\varepsilon = \alpha$ (in addition, Benassy has $\theta = 1$, corresponding to the center panel in Fig. 1). The next proposition puts these cases in perspective.

Proposition 5 *Assume A4. Then:*

(i) The equal-growth knowledge elasticity $\tilde{\nu}$ is below 1 in the Romer-Benassy case $\varepsilon = \alpha$;

(ii) a simple example where $\tilde{\nu}$ is above 1 is: $\theta = 1$, $\alpha > \max(\frac{\gamma L}{(1-\varepsilon)\rho+\gamma L}, \frac{\rho}{(1-\varepsilon)\gamma L+\rho})$; indeed,

(iii) for any $\nu > 0$ such that the social optimum has a steady state, the market economy has a steady state with $g_c > g_c^*$ if and only if $\phi(\varepsilon, \alpha) > \max(\nu + \frac{\nu-\beta}{\beta-(1-\theta)\nu}, \frac{\rho}{\gamma L})$.

Proof. See Appendix 8.3. ■

In the specific Romer case, in addition to $\varepsilon = \alpha$, $\nu = 1$; hence, by (i) of Proposition 5, $\tilde{\nu} < 1 = \nu$, and we get the Romer conclusion that $g_c < g_c^*$ unambiguously. Further, when $\varepsilon = \alpha$, the opposite inequality, $g_c > g_c^*$, arises if and only if the knowledge elasticity ν is considerably below 1; this confirms Benassy's conclusion. The interpretation is that in this case, the necessary amount of creative down-weighing requires ν to be considerably below 1, cf. (2.11). However, as implied by (iii) of Proposition 5, allowing for a separation of the substitution parameter, ε, from the share of capital, α, we have: For a sufficiently high profit factor $\phi(\varepsilon, \alpha)$, i.e., essentially a sufficiently high α, g_c does indeed end up larger than g_c^* even when ν equals 1 or is above 1.

Our model assumes that at the aggregate level there are always positive effects of research (since $\nu > 0$); a fundamental defect of the market economy is that the potential for monopoly profits does not adequately reflect these aggregate effects. Indeed, as Proposition 3 tells, resource allocation to research may be "perversely" related to the aggregate contribution of research as measured by ν. To understand what is involved, notice that the social planner sees the rate of return on the marginal investment in knowledge as consisting of two terms (disregarding, for the moment, capital gains):

$$\frac{1}{q}\left[\frac{\partial Y}{\partial A} + q\gamma(1-u)L\right] = (\nu u + 1 - u)\gamma L, \tag{5.2}$$

where q is the real shadow value of one unit of knowledge[24]. The term $\frac{\partial Y}{\partial A}$ represents the current net contribution to output ("net social surplus") and, given u, this is proportional to the knowledge elasticity ν, while the term $q\gamma(1-u)L$ represents a contribution to output in the future through higher productivity in the research sector. Hence, in the social planner's solution a larger ν induces more resources to R&D.

In contrast, the rate of return to investment in knowledge as seen by the private firm is determined by the attainable private profits:

$$\frac{\pi}{p} = \frac{1-\varepsilon}{1-\alpha}\alpha\gamma uL \tag{5.3}$$

from (4.17) and (4.20) (again we have, for the moment, disregarded capital gains). Monopoly power, as measured by $1-\varepsilon$, and the capital share, α, enter, but neither does ν (except possibly through u) nor the future contribution of more knowledge through higher productivity in the research sector.

The disconnection of the private return from the social return is the basis for the disorganization of the relation between ν and N_A in the market economy. This relation is now shaped only through the capital gains term $\dot{p}/p$, cf. (4.7), which in steady state takes the form $\dot{p}/p = (\nu-1)g_A$ (see (8.12)) where g_A is connected to g_c through

$$g_c = \nu g_A = \nu\gamma N_A. \tag{5.4}$$

This indicates why the sign of the effect of a larger ν on N_A is completely dependent on the desire for consumption smoothing, θ. By (5.4), a larger ν implies a larger multiplier on N_A, i.e., a potential for more growth. If the desire for consumption smoothing is intermediate (i.e., $\theta = 1$), then this potential gain in the future will be taken out as it is without adjusting N_A. If, however, the desire for consumption smoothing is low ($\theta < 1$), then the market mechanism will exploit this potential for enhanced growth more than one to one, by adjusting N_A upwards when ν becomes larger. On the other hand, if the desire for consumption smoothing is high ($\theta > 1$), we get a "perverse" reaction and part of the higher growth potential, generated by higher ν, will be taken out as more consumption today, that is by adjusting N_A downwards[25].

[24]The value of the marginal product of labor being equalized across sectors, we have $q\gamma A = (1-\alpha)Y/(uL)$ and $\partial Y/\partial A = \nu(1-\alpha)Y/A$. This explains the $\nu u\gamma L$ term in (5.2).

[25]In the social planner's solution there is a similar capital gains effect when $\theta > 1$, but it is always outdone by the positive effect of higher ν on the direct returns in (5.2).

As is suggested by the graphs in Fig. 1, private firms may have an incentive to allocate resources to R&D even in the limiting case $\nu = 0$, i.e., when R & D is socially useless. Since we have not seen this fact acknowledged in the literature, it seems worthwhile to give a formal statement[26].

Proposition 6 *Assume A4 and* $\nu = 0$. *Then there exists a unique steady state in the market economy; it has* $g_c = 0$,

$$N_A = \frac{(1-\varepsilon)\alpha\gamma L - \rho(1-\alpha)}{\gamma(1-\varepsilon\alpha)} > 0, \tag{5.5}$$

$g_A = \gamma N_A > 0$, *and* $\frac{\partial Y}{\partial K} = \alpha \frac{Y}{K} = \frac{1}{\varepsilon}(\rho + \delta)$. *In contrast, the unique steady state of the social optimum has* $g_c^* = N_A^* = g_A^* = 0$, *and* $\frac{\partial Y}{\partial K} = \alpha \frac{Y}{K} = \rho + \delta$.

Proof. See Appendix 8.3. ■

Though from a social point of view resources applied to R&D are wasted when the knowledge elasticity ν is zero, the firms bearing the research costs are able to recover them by taking advantage of their market power (granted by the patent legislation). As soon as a new specialized capital good is invented and supplied to the market, there is a demand for it whether $\nu > 0$ or $\nu = 0$ (due to the symmetry and the strict concavity of Y in x_i, $i = 1, 2, ..., A$). In the limiting case $\nu = 0$, the direct productive contribution of the new capital good is completely outweighed by the concomitant decrease in the productivity of the old capital goods, this decrease being interpreted as a result of higher complexity in a more specialized world.

6 Remarks on the empirics

In addition to the richer set of theoretical possibilities, the more general framework used in this paper allows better accordance with the empirical evidence on markups and the trend of the patent-R&D ratio than do the simpler models of Romer and Benassy. Consider first the patent-R&D ratio, that is, the number of new patents per year divided by R&D expenditures. Since the late fifties, in the US a systematic decline in this ratio has taken place (on average a fall at 3.5 % per year, see Griliches, 1989, and Kortum, 1993). In our model, the patent-R&D ratio is given

[26] This will be the only place in this paper where we allow $\nu = 0$, or equivalently $\eta = 0$, i.e., a violation of (P); because of this violation, one must be careful using previous derivations.

by $\dot{A}/(wN_A) = \gamma A/w$, and in a steady state this ratio will be decreasing over time if and only if the knowledge elasticity ν is larger than one (because, in a steady state, by (4.15), $g_w = \nu g_A$)[27]. But Romer-style models have $\nu = 1$, and they are therefore inconsistent with the observed fall in the patent-R&D ratio.

The markup is in Romer-style models implicitly given by the inverse of the capital share. The empirical evidence suggests that the capital share in output, α, is around 0.4 in the US (see, e.g., Cooley and Prescott (1995)) so that Romer-style models predict markups around 2.5! According to Norrbin (1993) and Basu (1996), however, markup estimations are between 1.05 and 1.40 in US industry. The present framework allows accordance with this, since we can choose a value for the substitution parameter ε in the interval $[0.70, 0.95]$.

According to Proposition 2 these observations indicate the following size relation between the equal-growth knowledge elasticity $\tilde{\nu}$, the profit factor $\phi(\varepsilon, \alpha)$, and the actual knowledge elasticity ν in US industry: $\tilde{\nu} < \frac{\phi(\varepsilon,\alpha)}{\varepsilon} \equiv \frac{(1-\varepsilon)\alpha}{(1-\alpha)\varepsilon} < 1 < \nu$. In particular, $\nu > \frac{\phi(\varepsilon,\alpha)}{\varepsilon}$ is an indication that there is "creative synergy", and $\nu > \tilde{\nu}$ indicates that there is too little R&D so that the growth rate is inefficiently low. This result is consistent with the empirical evidence presented by Jones and Williams (1998). These authors find that the optimal R&D investment is at least four times larger than the actual spending. Therefore, in this respect the prediction from the simple Romer framework (where $\nu = \frac{\phi(\varepsilon,\alpha)}{\varepsilon} = 1$) seems to point in the right direction. Our hint that $\nu > \frac{\phi(\varepsilon,\alpha)}{\varepsilon}$ is likely, strengthens the confidence in that prediction[28].

[27] To put it differently, the patent-R&D ratio falls when productivity in manufacturing increases faster than in R&D activity. This need not be a sign of exhaustion of technological opportunities. Rather than being something to worry about, it may, according to the present model, be a sign of high potentiality of new technical knowledge. (Of course, in reality the level of patenting lacks a lot in precision as an indicator of aggregate R&D successes inasmuch as many firms, at least outside the chemical and pharmaceutical industries, rely on other ways of protecting their innovations.)

[28] In Alvarez and Groth (2001b) it is shown that an active government needs two instruments to establish the optimal allocation. These instruments could be, first, a subsidy to buyers of capital services in order to eliminate distorting demand effects of monopoly pricing, and, second, a tax on - or subsidy to - monopoly profits, depending on the parameters (if the above parameter values are accepted, a subsidy is indeed called for).

7 Conclusions

According to the first generation of models of endogenous growth based on expanding product variety, the market economy unambiguously yield a too low level of R&D. However, disentangling returns to specialization from the market power parameter, Benassy (1998) found that this result arises due to the implicit choice of a relatively high value for the returns to specialization.

The present paper takes a step further, analyzing a model of expanding product variety with capital accumulation where returns to specialization, the monopolistic markup, and the capital share in final output are given by independent parameters. Previous expanding variety models can be seen as special cases. Among the implications of the general model are, first, that low returns to specialization are not necessary for the case with too much R&D in the market economy to arise. Indeed, whatever the returns to specialization, the decisive factor is the implicit presence of enough negative disaggregate externalities of increased specialization.

Second, we show the usefulness of placing the expanding variety models, with the natural parameter separations included, on a similar footing as the vertical innovation models by decomposing the repercussions of innovations into different types of "effects": an "intertemporal spillover", a "direct surplus effect", and an externality on other producers. In this way, there seems to be much less asymmetry between the horizontal and vertical innovation models than hitherto recognized. Further, we find that even when R&D is socially useless (i.e., the externality on other producers is negative and strong enough to outweigh the other effects), the market equilibrium may have resources allocated to R&D.

Third, the separation of the markup from the share of capital (or more generally the share of intermediate goods) allows the markup to take values more in line with the empirical evidence. Thereby a weakness of the Romer-style expanding variety models is avoided since these models (where the markup is directly linked to the share of intermediate goods in manufacturing) lead to unrealistically high values of the markup. In addition, allowing returns to specialization to be higher than implicit in Romer-style models permits the patent-R&D ratio to decrease over time, in accordance with the falling patent-R&D ratio observed in the data.

An extension of the analysis would be to apply a similar separation of parameters to the class of "semi-endogenous" increasing-variety growth models (cf. Jones 1995) where the R&D equation is $\dot{A} = \gamma N_A A^{\mu}$ with $\mu < 1$ instead of $\mu = 1$ as above. Then, comparing the market economy and the social optimum with respect to resource allocation and the *level* of the growth path (rather than growth *rates*) would be an interesting undertaking. Further, again following Jones (1995), a duplication externality could be added so that the right hand side of the R&D equation reads $\gamma N_A \bar{N}_A^{\lambda-1} A^{\mu}$, $0 < \lambda < 1$, where $\bar{N}_A$ is external to the individual research firm, and $N_A = \bar{N}_A$ in equilibrium. Introducing capital as an input in R&D and using more general technology specifications would also be natural[29]. A well-known, more fundamental weakness of first generation increasing-variety models is that complete obsolescence of old intermediate inputs never occurs. Extending the framework by adding full-fledged "creative destruction" (along the lines of Jones and Williams 2000) and combining with vertical innovation (as in Howitt 1999) is at the same time a continuation and a widening of the focus of the present paper on conflicting macro effects of the market organization of R&D.

8 Appendix

8.1 Technology. The social optimum

Proof of Lemma 1. (i) Consider a steady state with $g_A = \bar{g}_A$. Then, from (2.4), N_Y is constant. By definition of a steady state, $Y > 0$; hence, from (2.7), $N_Y > 0$. Therefore $u \equiv N_Y/L$ is constant, $u \in \left(0, 1 - \frac{\bar{g}_A}{\gamma L}\right]$, and $0 \leq \bar{g}_A = \gamma(1-u)L < \gamma L$. (ii) Y/K constant implies $g_Y = g_K$ which is constant in a steady state. Then, by (2.3), cL/K is constant, hence $g_c = g_K$. From (2.7), the common growth rate, $\bar{g}$, of c, K, and Y is $\frac{\eta}{1-\alpha}\bar{g}_A$. ■

Proof of Proposition 1. Let $\alpha, \varepsilon, \eta$, $\theta, \rho, \delta, \gamma$, and L be given, satisfying (P), and consider a given $A_0 > 0$. Let $\nu \equiv \frac{\eta}{1-\alpha}$. Choose any $\bar{g} \in [0, \nu\gamma L)$. We will construct a path SSP $= (K, A, C, Y, N_Y, N_A)_{t=0}^{\infty}$ with $g_C = g_Y = g_K = \bar{g}$ and with the path for A "starting" from the given A_0. Let $N_Y = L - N_A = u(\bar{g})L$ where $u(\bar{g}) \equiv 1 - \frac{1}{\nu\gamma L}\bar{g} > 0, u' < 0$. Then, by (2.4), $g_A = \gamma(1-u(\bar{g}))L = \frac{1}{\nu}\bar{g}$ and $Y = K^{\alpha}(A^{\nu}u(\bar{g})L)^{1-\alpha}$. Now, $\partial Y/\partial K = \alpha Y/K$. Choose any constant $z \geq (\bar{g}+\delta)/\alpha$. We can then show that our SSP will have an output-capital ratio $Y/K = z$, a net marginal product of capital $\partial Y/\partial K - \delta = \alpha z - \delta \geq \bar{g}$, and $g_Y = g_K = \bar{g}$ if and only

[29] An elasticity of substitution between X and N_Y higher than $1/(1-\varepsilon)$ in (2.1) (where it is 1) could allow multiple steady states and underdevelopment traps to occur, cf. Ciccone and Matsuyama (1996).

if the investment-output ratio I/Y takes the value $(\bar{g}+\delta)/z \in (0,\alpha]$. Indeed, from (2.3) and the definition $I \equiv Y - C$ we have $g_K = \frac{I}{K} - \delta = \frac{I}{Y}\frac{Y}{K} - \delta = \frac{I}{Y}z - \delta = \bar{g}$ $\Leftrightarrow \frac{I}{Y} = (\bar{g}+\delta)/z$; in particular, $z \geq (\bar{g}+\delta)/\alpha \Leftrightarrow 0 < (\bar{g}+\delta)/z \leq \alpha$. Moreover, defining $\tilde{k} \equiv K/(A^{\nu}\bar{u}L)$, (2.3) and the identity $C \equiv cL$ imply

$$\dot{\tilde{k}} = \tilde{k}^{\alpha} - \frac{c}{A^{\nu}u} - (\nu g_A + \delta)\tilde{k}. \tag{8.1}$$

Along our SSP we have $\nu g_A = \bar{g}, u = u(\bar{g})$, and $Y/K = \tilde{k}^{\alpha-1} = z$, a constant. The last equality implies $\dot{\tilde{k}} = 0$. Then (8.1) gives

$$c = c(\tilde{k},\bar{g})u(\bar{g})A_0^{\nu}e^{\bar{g}t}, \tag{8.2}$$

where $c(\tilde{k},\bar{g}) \equiv \tilde{k}^{\alpha} - (\bar{g}+\delta)\tilde{k}$. Let $\tilde{k}_{GR}$ denote the golden rule capital intensity satisfying $\alpha\tilde{k}_{GR}^{\alpha-1} = \bar{g}+\delta$. In view of $\tilde{k}^{\alpha-1} = z \geq (\bar{g}+\delta)/\alpha$ we have $\tilde{k} \leq \tilde{k}_{GR}$, $0 < c(\tilde{k},\bar{g}) \leq max_{\tilde{k}}c(\tilde{k},\bar{g}) = \tilde{k}_{GR}^{\alpha} - (\bar{g}+\delta)\tilde{k}$. From this we see that the so constructed SSP is a dynamically efficient steady state path with A "starting" from the given A_0, with $g_C = g_Y = g_K = \bar{g}$, and with $Y/K = z$. Further, $\frac{\partial c}{\partial \tilde{k}}(\tilde{k},\bar{g}) > 0$ when $\tilde{k} < \tilde{k}_{GR}$; and, considering $\tilde{k} = z^{\frac{1}{\alpha-1}}$, given $\bar{g}$ and t, c is at maximum when $z = (\bar{g}+\delta)/\alpha$, and, since $\frac{\partial \tilde{k}}{\partial z} < 0$, c is decreasing in z when $z > (\bar{g}+\delta)/\alpha$. Finally, (8.2) shows that c is increasing in A_0. ■

Social planning problem (3.1). In order to obtain concavity of the maximized Hamiltonian, we introduce the transformations $\tilde{A} \equiv A^{\frac{\eta}{1-\alpha}}$ and $\tilde{\gamma} \equiv \frac{\eta}{1-\alpha}\gamma$. Then $Y = K^{\alpha}(\tilde{A}N_Y)^{1-\alpha}$, and $\dot{\tilde{A}} = \tilde{\gamma}(L-N_Y)\tilde{A}$. The current value Hamiltonian for the social planning problem of Section 3 becomes

$$H = \frac{c^{1-\theta}-1}{1-\theta} + \lambda_1[K^{\alpha}(\tilde{A}N_Y)^{1-\alpha} - \delta K - cL] + \lambda_2\tilde{\gamma}(L-N_Y)\tilde{A},$$

where λ_1 and λ_2 are the shadow prices of the state variables K and $\tilde{A}$, respectively. Necessary conditions for an interior solution are that for all $t \geq 0$:

$$c^{-\theta} = \lambda_1 L, \tag{8.3}$$

$$\lambda_1(1-\alpha)\frac{Y}{\tilde{A}N_Y} = \lambda_2\tilde{\gamma}, \tag{8.4}$$

$$\dot{\lambda}_1 = \rho\lambda_1 - \lambda_1(\frac{\partial Y}{\partial K} - \delta), \tag{8.5}$$

$$\dot{\lambda}_2 = \rho\lambda_2 - \lambda_1\frac{\partial Y}{\partial \tilde{A}} - \lambda_2\tilde{\gamma}(L-N_Y), \tag{8.6}$$

$$\lim_{t\to\infty}\lambda_1 e^{-\rho t}K = 0, \lim_{t\to\infty}\lambda_2 e^{-\rho t}\tilde{A} = 0. \tag{8.7}$$

Log-differentiating (8.3) wrt. t and using (8.5) gives

$$g_c \equiv \frac{\dot{c}}{c} = \frac{1}{\theta}(\frac{\partial Y}{\partial K} - \delta - \rho) = \frac{1}{\theta}(\alpha\frac{y}{k} - \delta - \rho) = \frac{1}{\theta}(\alpha\tilde{k}^{\alpha-1} - \delta - \rho), \qquad (8.8)$$

where $k \equiv \frac{K}{N_Y} \equiv \frac{K}{uL}$, $y \equiv \frac{Y}{uL} = \tilde{A}^{1-\alpha}k^{\alpha}$, and $\tilde{k} \equiv k/\tilde{A}$. Since in a steady state, by definition, g_c is constant, in view of (8.8), $\frac{y}{k}$ and $\tilde{k}$ are also constant.

Define $\tilde{q} \equiv \lambda_2/\lambda_1$. From (8.4)

$$\tilde{q} = \frac{(1-\alpha)Y}{\tilde{\gamma}\tilde{A}N_Y} = \frac{1-\alpha}{\tilde{\gamma}}\tilde{k}^{\alpha}. \qquad (8.9)$$

Hence, in steady state

$$\frac{\dot{\tilde{q}}}{\tilde{q}} = 0. \qquad (8.10)$$

But, by definition of $\tilde{q}$, $\dot{\tilde{q}}/\tilde{q} = (\dot{\lambda}_2/\lambda_2) - (\dot{\lambda}_1/\lambda_1)$, hence, from (8.5), and (8.6),

$$\frac{\dot{\tilde{q}}}{\tilde{q}} = \frac{\partial Y}{\partial K} - \delta - \tilde{\gamma}L. \qquad (8.11)$$

(8.10), (8.11), and the definition of $\tilde{\gamma}$ imply $\frac{\partial Y}{\partial K} = \frac{\eta}{1-\alpha}\gamma L + \delta$, i.e., (3.2) in the text. Inserting this into (8.8) gives (3.3) in the text.

A path $(c, N_Y, K, \tilde{A})_{t=0}^{\infty}$, which satisfies the first order and transversality conditions (8.3), (8.4), (8.5), (8.6), and (8.7), and which approaches the steady state, is our candidate for an optimal solution. To ensure that such a path is really an optimal solution, we check, first, whether the Hamiltonian is jointly concave in the state variables K and $\tilde{A}$ after the controls c and N_Y have been substituted by their maximizing values from (8.3) and (8.4), respectively. We find $H = B + \lambda_1\tilde{\gamma}LK + \lambda_2\tilde{\gamma}L\tilde{A}$, where the term B does not depend on K or $\tilde{A}$. Clearly this function is concave.

The second aspect to check is whether our candidate path $(c, N_Y, K, \tilde{A})_{t=0}^{\infty}$ satisfies a *sufficient* transversality condition. In view of (8.3) and (8.4), $\lambda_1 > 0$ and $\lambda_2 > 0$ for all $t \geq 0$. Hence, by (8.7), the path $(c, N_Y, K, \tilde{A})_{t=0}^{\infty}$ satisfies

$$\lim_{t\to\infty}\left[\lambda_1 e^{-\rho t}(\hat{K} - K) + \lambda_2 e^{-\rho t}(\widehat{\tilde{A}} - \tilde{A})\right] \geq 0,$$

for all feasible paths $(\hat{c}, \hat{N}_Y, \hat{K}, \widehat{\tilde{A}})_{t=0}^{\infty}$. The conclusion is that, by Arrow's sufficiency theorem, our candidate path $(c, N_Y, K, \tilde{A})_{t=0}^{\infty}$ *is* an optimal solution, see Seierstad and Sydsæter (1987, p. 236).

Finally, we will show that the steady state is saddlepoint stable. Let $z \equiv Y/K$ and $\bar{c} \equiv cL/K$. Then, from the first order conditions (8.3), (8.4), (8.5), (8.6), and the dynamic constraints of the problem (3.1), we get the differential equations

$$\begin{aligned}
\dot{z} &= \left[(\alpha-1)z + \frac{1-\alpha}{\alpha}(\tilde{\gamma}L+\delta)\right] z, \\
\dot{\bar{c}} &= \left[\frac{\alpha-\theta}{\theta}z + \bar{c} + \frac{\theta-1}{\theta}\delta - \frac{\rho}{\theta}\right]\bar{c}, \\
\dot{u} &= \left[-\hat{c} + \tilde{\gamma}Lu + \frac{1-\alpha}{\alpha}(\tilde{\gamma}L+\delta)\right] u.
\end{aligned}$$

The Jacobian evaluated in the steady state is triangular and has the eigenvalues $\rho_1 = (\partial\dot{z}/\partial z)^* = (\alpha-1)z^* < 0$, $\rho_2 = (\partial\dot{\bar{c}}/\partial\bar{c})^* = \bar{c}^* > 0$, and $\rho_3 = (\partial\dot{u}/\partial u)^* = \tilde{\gamma}Lu^* > 0$. The three variables, $z, \bar{c}$, and u, are jump variables, but the initial conditions for z and u are not independent. Given the predetermined variable $\bar{k} \equiv K/(\tilde{A}L)$, we have $u = \bar{k}z^{\frac{1}{1-\alpha}}$, from the production function $Y = K^{\alpha}(\tilde{A}N_Y)^{1-\alpha}$. Hence, with two free initial conditions and exactly two positive eigenvalues, uniqueness of a convergent solution (that is, saddlepoint stability) holds generically.

8.2 The market economy

Steady state. In an interior equilibrium, by (4.17), the market value of a patent grows according to

$$g_p = \alpha g_{\tilde{k}} + \frac{\eta+\alpha-1}{1-\alpha}g_A. \tag{8.12}$$

Inserting (4.20), (4.17), (4.15), and (8.12) into (4.7), using the fact that $\gamma uL = \gamma L - g_A$, from (2.4), gives the market interest rate as

$$r = \frac{1-\varepsilon}{1-\alpha}\alpha(\gamma L - g_A) + (\frac{\eta}{1-\alpha} - 1)g_A + \alpha g_{\tilde{k}}. \tag{8.13}$$

In a steady state, by definition c grows at a constant rate, hence, from (4.12), r is constant. Therefore, Y/K and $\tilde{k}$ are constant in view of (4.16), implying $g_y = g_c = g_k = \frac{\eta}{1-\alpha}g_A$ from Lemma 1 and the definition of $\tilde{k}$. By (4.12) this implies

$$g_A = \frac{1-\alpha}{\eta\theta}(r-\rho). \tag{8.14}$$

An interior steady state has $u < 1$, hence (8.13) holds. Inserting the steady state condition $g_{\tilde{k}} = 0$ into (8.13) gives a "growth possibility curve" and combining this with the "growth preference curve", (8.14), leads to

$$g_A = \frac{(1-\varepsilon)\alpha\gamma L - (1-\alpha)\rho}{1-\varepsilon\alpha-(1-\theta)\eta}. \tag{8.15}$$

Multiplying by $\frac{\eta}{1-\alpha}$ gives the formula (4.18). Combining (4.12) and (4.16) gives (4.19).

The interiority of the steady state means, by definition, $0 < u < 1$ where

$$u = \frac{[1-\alpha-(1-\theta)\eta]\,\gamma L + (1-\alpha)\rho}{[1-\varepsilon\alpha-(1-\theta)\eta]\,\gamma L}, \tag{8.16}$$

from (8.15) and (2.4) with $N_A = (1-u)L$. Firstly, consider the "normal case" defined by the denominator of g_A being positive, that is

$$1-\varepsilon\alpha > (1-\theta)\eta. \tag{8.17}$$

Given (8.17), $u < 1$ holds if and only if

$$\rho < \frac{1-\varepsilon}{1-\alpha}\alpha\gamma L, \tag{A4}$$

and $u > 0$ if and only if

$$\rho > \left[(1-\theta)\frac{\eta}{1-\alpha} - 1\right]\gamma L. \tag{8.18}$$

However, for the steady state to be an equilibrium, the transversality condition (4.13) should be satisfied, and this requires u to have finite distance from 0. Indeed, $v \equiv k + \frac{pA}{L}$, and in steady state $g_k = g_c = \frac{\eta}{1-\alpha}g_A = g_p + g_A$, from (8.12); hence $g_v = g_c$. It follows that the steady state satisfies (4.13) if and only if the steady state has

$$r > g_c. \tag{8.19}$$

By (4.12), $r = \theta g_c + \rho$, and if $\theta \geq 1$, (8.19) holds automatically. Suppose, on the contrary, $\theta < 1$. The value of η for which $\theta g_c + \rho = g_c$ is, using $g_c = \nu g_A$, where g_A is given in (8.15),

$$\bar{\eta} \equiv \frac{(1-\varepsilon\alpha)(1-\alpha)\rho}{(1-\theta)(1-\varepsilon)\alpha\gamma L}. \tag{8.20}$$

Now, $0 < \frac{\partial r}{\partial \eta} = \theta\frac{\partial g_c}{\partial \eta} < \frac{\partial g_c}{\partial \eta}$ in this case, since, as shown below, $\frac{\partial g_c}{\partial \eta} > 0$. Therefore, (8.19) holds if and only if $\eta < \bar{\eta}$, and we have to strengthen the requirement $u > 0$ by substituting

$$\rho > \frac{1-\varepsilon}{1-\alpha}\alpha\gamma L\frac{1-\theta}{1-\varepsilon\alpha}\eta \tag{A3}$$

for (8.18). By (8.16), A4 together with A3 is equivalent to $\frac{1-\alpha}{1-\varepsilon\alpha} < u < 1$.

It follows that A3 guarantees $r > g_c$. Therefore, A3 gurantees the transversality condition of the household. Combining (4.12) and (4.16) gives (4.19).

Proof of Lemma 2 *(i)* $\frac{(1-\varepsilon)\alpha}{1-\varepsilon\alpha} = \frac{\alpha-\varepsilon\alpha}{1-\varepsilon\alpha} < 1$; hence, A1 $\Rightarrow$ A3. *(ii)* A3 and A4 $\Rightarrow \frac{(1-\varepsilon)\alpha(1-\theta)}{(1-\alpha)(1-\varepsilon\alpha)}\eta < \frac{\rho}{\gamma L} < \frac{1-\varepsilon}{1-\alpha}\alpha$; hence, $(1-\theta)\eta < 1-\varepsilon\alpha$. (iii) This is obvious from (P). ■

Lemma A Assume A3 and (8.17). Then (8.18) holds.

Proof. (8.17) $\Rightarrow (1-\theta)\frac{\eta}{1-\alpha}(1-\varepsilon\alpha+\varepsilon\alpha-\alpha) < 1-\varepsilon\alpha \Rightarrow (1-\theta)\frac{\eta}{1-\alpha}\left[1-\frac{(1-\varepsilon)\alpha}{1-\varepsilon\alpha}\right] < 1 \Rightarrow (1-\theta)\frac{\eta}{1-\alpha} - 1 < \frac{(1-\varepsilon)\alpha}{1-\varepsilon\alpha}(1-\theta)\frac{\eta}{1-\alpha} < \frac{\rho}{\gamma L}$, by A3. ■

Corollary 2 *Assume A3 and A4. Then (8.18) holds.*

Proof. By Lemma 2, A3 and A4 imply (8.17). Now use Lemma A. ■

Varying the parameters. Assume A3 and A4. Then, by Lemma 2 and Lemma A, (8.17) and (8.18) hold. From (4.18) we get, after some manipulations,

$$\frac{\partial g_c}{\partial \eta} = \frac{(1-\varepsilon\alpha)\left[(1-\varepsilon)\alpha\gamma L - (1-\alpha)\rho\right]}{\left[1-\varepsilon\alpha-(1-\theta)\eta\right]^2(1-\alpha)} > 0,$$

by A4;

$$\frac{\partial g_c}{\partial \varepsilon} = \frac{\left[\gamma L((1-\theta)\eta-(1-\alpha)) - (1-\alpha)\rho\right]}{\left[1-\varepsilon\alpha-(1-\theta)\eta\right]^2(1-\alpha)}\alpha\eta < 0,$$

by (8.18); and, when η is given,

$$\frac{\partial g_c}{\partial \alpha} = \frac{\frac{1-\varepsilon}{1-\alpha}\gamma L\left[1-\varepsilon\alpha-(1-\theta)\eta\right] + \varepsilon\left[(1-\varepsilon)\alpha\gamma L-(1-\alpha)\rho\right]}{\left[1-\varepsilon\alpha-(1-\theta)\eta\right]^2(1-\alpha)}\eta > 0$$

by (8.17) and A4 applied to the first and the second square bracket, respectively, in the numerator.

Dynamics. The equations describing the dynamics of the market economy can be reduced to three differential equations in $z \equiv Y/K$, $\bar{c} \equiv cL/K$, and u. The determinant of the Jacobian evaluated at the steady state can be shown to be negative. Hence, either there are three eigenvalues with negative real part or one negative eigenvalue and two with non-negative real part. A sufficient though not necessary condition for the last case to obtain, is that $\nu \leq (1-\varepsilon\alpha)/(1-\alpha)^2$ *or* $\theta = 1$ (see Alvarez and Groth 2001a). With $\alpha = .4$ and $\varepsilon < .95$, we have $(1-\varepsilon\alpha)/(1-\alpha)^2 > 1.7$, a number that seems beyond realistic values of ν. In view of the boundary condition, $u = [K/(A^{\nu}L)]\, z^{\frac{1}{1-\alpha}}$, generically, this implies uniqueness of the convergent solution (that is, saddlepoint stability) at least within the empirically relevant domain of the parameter space.

8.3 Comparison

Proof of Proposition 2. Assume A1, A3, and $\rho < \frac{1-\varepsilon}{1-\alpha}\alpha\gamma L$ (i.e., A4). Existence, uniqueness, and the formulas for g_c^* and g_c, respectively, were shown in Sections 3 and 4 and Appendix 8.2. Taking ν as an independent parameter (so that $\eta = (1-\alpha)\nu$) we have

$$\frac{\partial g_c^*}{\partial \nu} = \frac{\gamma L}{\theta} > 0,$$

and

$$\begin{aligned}
\frac{\partial g_c}{\partial \nu} &= \frac{\partial g_c}{\partial \eta}\frac{\partial \eta}{\partial \nu} = \frac{\partial g_c}{\partial \eta}(1-\alpha) > 0 \qquad \text{(from Appendix 9.2)},\\
\frac{\partial g_c}{\partial \alpha} &= (1-\varepsilon)\frac{[1-(1-\theta)\nu]\,\gamma L + \rho}{[1-\varepsilon\alpha-(1-\theta)(1-\alpha)\nu]^2}\nu > 0 \text{ when } \nu \text{ is given},\\
\frac{\partial g_c}{\partial \varepsilon} &> 0 \qquad \text{(from Appendix 9.2)},
\end{aligned}$$

where the last two inequalities follow from the Corollary to Lemma A. ■

Proof of Proposition 3. By (2.4), $N_A^* = g_A^*/\gamma$ and $N_A = g_A/\gamma$. Now, given A1 and A4, the conclusions follow from the qualitative features of g_A and g_A^* as functions of ν, listed in Table 1 (derived on the basis of Proposition 2). ■

Proof of Proposition 4. Assume A4 and let $\beta \equiv \frac{\rho}{\gamma L}$.

(i) If an *equal-growth knowledge elasticity* $\tilde{\nu}$ exists, it must, by definition, satisfy A1; if it did not, we know from Section 3 that the social optimum would have no steady state. Therefore, consider a ν such that A1 holds. In view of Lemma 2, A1 and A4 imply A3 so that for this ν both the social optimum and the market economy have a unique steady state, by Proposition 2, and

$$\frac{g_c^*}{g_c} = \frac{[1-\varepsilon\alpha-(1-\theta)(1-\alpha)\nu]\,(\nu-\beta)}{[(1-\varepsilon)\alpha-(1-\alpha)\beta]\,\theta\nu} \equiv \varsigma(\nu;\theta). \tag{8.21}$$

$0<\theta<1$	$\theta \geq 1$
$g_A^* = \begin{cases} 0 \text{ if } 0<\nu\leq \frac{\rho}{\gamma L}, \\ \frac{1}{\theta}(\gamma L - \frac{\rho}{\nu}) > 0 \text{ if } \frac{\rho}{\gamma L} < \nu < \frac{\rho}{(1-\theta)\gamma L}, \\ \text{undefined if } \nu \geq \frac{\rho}{(1-\theta)\gamma L}. \end{cases}$ $\frac{\partial g_A^*}{\partial \nu} > 0$ if $\frac{\rho}{\gamma L} < \nu < \frac{\rho}{(1-\theta)\gamma L}$. $\lim_{\nu\to\frac{\rho}{(1-\theta)\gamma L}} g_A^* = \gamma L$.	$g_A^* = \begin{cases} 0 \text{ if } 0<\nu\leq \frac{\rho}{\gamma L}, \\ \frac{1}{\theta}(\gamma L - \frac{\rho}{\nu}) > 0 \text{ if } \nu > \frac{\rho}{\gamma L}. \end{cases}$ $\frac{\partial g_A^*}{\partial \nu} > 0$ if $\nu > \frac{\rho}{\gamma L}$. $\lim_{\nu\to\infty} g_A^* = \frac{1}{\theta}\gamma L \leq \gamma L$
$g_A = \begin{cases} \frac{(1-\varepsilon)\alpha\gamma L-(1-\alpha)\rho}{1-\varepsilon\alpha-(1-\theta)(1-\alpha)\nu} > 0 \text{ if } 0<\nu<\frac{(1-\varepsilon\alpha)\rho}{(1-\varepsilon)\alpha(1-\theta)\gamma L}, \\ \text{undefined if } \nu \geq \frac{(1-\varepsilon\alpha)\rho}{(1-\varepsilon)\alpha(1-\theta)\gamma L}. \end{cases}$ $\frac{\partial g_A}{\partial \nu} > 0$ if $0<\nu<\frac{(1-\varepsilon\alpha)\rho}{(1-\varepsilon)\alpha(1-\theta)\gamma L}$. $\lim_{\nu\to 0} g_A = \frac{(1-\varepsilon)\alpha\gamma L-(1-\alpha)\rho}{1-\varepsilon\alpha} \in (0,\gamma L)$, $\lim_{\nu\to\frac{(1-\varepsilon\alpha)\rho}{(1-\varepsilon)\alpha(1-\theta)\gamma L}} g_A = \frac{(1-\varepsilon)\alpha}{1-\varepsilon\alpha}\gamma L < \gamma L$.	$g_A = \frac{(1-\varepsilon)\alpha\gamma L-(1-\alpha)\rho}{1-\varepsilon\alpha-(1-\theta)(1-\alpha)\nu} > 0$ for all $\nu > 0$; $\frac{\partial g_A}{\partial \nu} \begin{cases} = 0 \text{ if } \theta = 1, \\ < 0 \text{ if } \theta > 1. \end{cases}$ $\lim_{\nu\to 0} g_A = \frac{(1-\varepsilon)\alpha\gamma L-(1-\alpha)\rho}{1-\varepsilon\alpha} \in (0,\gamma L)$ $\lim_{\nu\to\infty} g_A = \begin{cases} \frac{(1-\varepsilon)\alpha\gamma L-(1-\alpha)\rho}{1-\varepsilon\alpha} \text{ if } \theta = 1, \\ 0 \text{ if } \theta > 1. \end{cases}$

Table 1: Properties of the knowledge growth rates as functions of the knowledge elasticity (A4 is presupposed).

The condition $\varsigma(\nu;\theta) = 1$ implies the quadratic equation

$$Q(v) \equiv (1-\theta)\nu^2 - [1+\beta+(1-\theta)\phi(\varepsilon,\alpha)]\,\nu + \beta(1+\phi(\varepsilon,\alpha)) = 0,$$

where $\phi(\varepsilon,\alpha)$ is the profit factor, i.e., $\phi(\varepsilon,\alpha) \equiv \frac{1-\varepsilon}{1-\alpha}\alpha > 0$. For $\theta \neq 1$ the roots are

$$\binom{\nu_2}{\nu_1} = \frac{1+\beta+(1-\theta)\phi(\varepsilon,\alpha) \pm \sqrt{D}}{2(1-\theta)}, \tag{8.22}$$

where

$$\begin{aligned} D &\equiv [1+\beta+(1-\theta)\phi(\varepsilon,\alpha)]^2 - 4(1-\theta)\beta(1+\phi(\varepsilon,\alpha)) \\ &= [1-\beta+(1-\theta)\phi(\varepsilon,\alpha)]^2 + 4\theta\beta > 4\theta\beta > 0. \end{aligned}$$

Case 1: $\theta < 1$. From (8.22) follows $0 < \nu_1 < \nu_2$; we have $Q(\nu) < 0$ for $\nu_1 < \nu < \nu_2$ and $Q(\nu) > 0$ for $\nu < \nu_1$ and $\nu > \nu_2$. Since $Q(\frac{\beta}{1-\theta}) = \frac{-\theta\beta}{1-\theta} < 0$, $\nu_1 < \frac{\beta}{1-\theta} < \nu_2$. Therefore, ν_1 satisfies A1, but ν_2 does not satisfy A1 and can be discarded. Hence, the unique equal-growth knowledge elasticity is $\nu_1 \in (0, \frac{\beta}{1-\theta})$.

Case 2: $\theta > 1$. Now $D > [1+\beta+(1-\theta)\phi(\varepsilon,\alpha)]^2$ so that

$$1+\beta+(1-\theta)\phi(\varepsilon,\alpha) \pm \sqrt{D} \gtrless 0,$$

respectively. Hence, $\nu_2 < 0 < \nu_1$, and again ν_2 can be discarded. We have $Q(\nu) > 0$ for $\nu_1 < \nu < \nu_2$ and $Q(\nu) < 0$ for $\nu < \nu_2$ and $\nu > \nu_1$. Again the unique equal-growth knowledge elasticity is $\nu_1 > 0$.

Case 3: $\theta = 1$. In this case $Q(\nu) = 0$ has one root

$$\nu_1 (= \nu_2) = \frac{\beta}{1+\beta}(1+\phi(\varepsilon,\alpha)), \tag{8.23}$$

and $Q(\nu) \gtrless 0$ for $\nu \lesseqgtr \nu_1$, respectively.

In all three cases the unique equal-growth knowledge elasticity is $\tilde{\nu} = \nu_1$.

(ii) That $\tilde{\nu} < \frac{\beta}{1-\theta}$ when $\theta < 1$ (i.e., A1 holds), was shown above. If on the other hand $\theta \geq 1$, then, by choosing α sufficiently close to 1, A1 and A4 are still satisfied, and g_A comes arbitrarily close to $\frac{1}{\theta}\gamma L$, the limiting value of g_A^* for $\nu \to \infty$. That $\tilde{\nu} > \beta$ can be seen in the following way. By A4, $g_c > 0$; hence $\varsigma(\tilde{\nu};\theta) = 1 \Rightarrow g_c^* = g_c > 0$, i.e., $\tilde{\nu} > \beta$ from Proposition 2. To show that $\tilde{\nu} < \phi(\varepsilon,\alpha)$, consider

$$\begin{aligned} Q(\phi(\varepsilon,\alpha)) &= (1-\theta)(\phi(\varepsilon,\alpha))^2 - (1+\beta)\phi(\varepsilon,\alpha) - (1-\theta)(\phi(\varepsilon,\alpha))^2 + \beta(1+\phi(\varepsilon,\alpha)) \\ &= \beta - \phi(\varepsilon,\alpha) < 0, \quad \text{by A4.} \end{aligned}$$

Hence, whether case 1, case 2, or case 3 above is true, we have $\nu_1 < \phi(\varepsilon, \alpha)$.

(iii) Consider a ν such that A1 holds. In view of Lemma 2, A1 and A4 imply A3 so that for this ν both the social optimum and the market economy have a unique steady state, by Proposition 2. It is seen from Table 1 that $\nu \overset{\leq}{\underset{>}{=}} \tilde{\nu} \Rightarrow g_A \overset{\geq}{\underset{<}{=}} g_A^*$, respectively. Hence, in view of $\eta \equiv (1-\alpha)\nu$ and (ii) of Lemma 1, $\nu \overset{\leq}{\underset{>}{=}} \tilde{\nu} \Rightarrow g_c \overset{\geq}{\underset{<}{=}} g_c^*$, respectively. ■

Proof of Proposition 5. Assume A4. (i) If $\varepsilon = \alpha$, then $\phi(\varepsilon, \alpha) = \alpha$, and by Proposition 4 we have $\tilde{\nu} < \alpha < 1$. (ii) Let $\theta = 1$. Then, from (8.23), $\tilde{\nu} = \nu_1 = \frac{\rho}{\rho+\gamma L}\frac{1-\varepsilon\alpha}{1-\alpha} \overset{\geq}{\underset{<}{=}} 1$ for $\alpha \overset{\geq}{\underset{<}{=}} \frac{\gamma L}{(1-\varepsilon)\rho+\gamma L}$, respectively. But A4 holds if and only if $\alpha > \frac{\rho}{(1-\varepsilon)\gamma L+\rho}$. (iii) Consider a $\nu > 0$ such that the social optimum has a steady state. Then, from Section 3 we know that if $\theta < 1$, then $\nu < \frac{\rho}{(1-\theta)\gamma L}$, i.e., A1 is satisfied. In view of Lemma 2, A1 and A4 imply A3 so that for this ν both the social optimum and the market economy have a unique steady state, by Proposition 2. Now, suppose $\nu > \frac{\rho}{\gamma L}$. Then, $\nu + \frac{\nu\gamma L-\rho}{\rho-(1-\theta)\nu\gamma L} > \nu > \frac{\rho}{\gamma L}$, and straightforward calculation from Proposition 2 gives

$$g_c > g_c^* \Leftrightarrow \frac{1-\varepsilon}{1-\alpha}\alpha > \nu + \frac{\nu\gamma L - \rho}{\rho - (1-\theta)\nu\gamma L},$$

where $\frac{1-\varepsilon}{1-\alpha}\alpha \equiv \phi(\varepsilon, \alpha)$. If instead, $\nu \leq \frac{\rho}{\gamma L}$, then $\nu + \frac{\nu\gamma L-\rho}{\rho-(1-\theta)\nu\gamma L} \leq \nu \leq \frac{\rho}{\gamma L}$. By Proposition 2, $g_c^* = 0$; but $g_c > 0$ if and only if $\frac{1-\varepsilon}{1-\alpha}\alpha > \frac{\rho}{\gamma L}$, proving (iii). ■

Proof of Proposition 6. Since $\nu \equiv \frac{\eta}{1-\alpha} = 0$ violates (P), we must start from basics or at least use only relations not depending on $\nu > 0$. From $k \equiv \tilde{k}A^\nu$ follows $k = \tilde{k}$ when $\nu = 0$. Then, in a steady state, by (4.12) and (4.16), $g_k = 0$. Since in a steady state, g_A is constant, u is constant; then, by (2.7), $g_Y = 0$, implying $\frac{Y}{K}$ constant and, by (2.3), C/K constant so that $g_c = g_C = g_K = g_k = 0$. From (4.12) and (8.13) now follows that in the market economy $\rho = r = \frac{1-\varepsilon}{1-\alpha}\alpha(\gamma L - g_A) - g_A$, and this, together with (2.4), implies $g_A = \frac{(1-\varepsilon)\alpha\gamma L-\rho(1-\alpha)}{1-\varepsilon\alpha} > 0$, given A4. Then, again by (2.4), $N_A = \frac{(1-\varepsilon)\alpha\gamma L-\rho(1-\alpha)}{\gamma(1-\varepsilon\alpha)} > 0$, confirming (5.5). With $r = \rho$, (4.16) gives $\frac{\partial Y}{\partial K} = \alpha\frac{Y}{K} = \frac{1}{\varepsilon}(\rho+\delta)$.

As to the social planner's problem (3.1), when $\nu = 0$, then $\eta = 0$, and increasing A is of no use (it is a waste of resources in fact); the problem reduces to a standard one-sector Ramsey problem with no technical progress. Hence, in steady state, $g_c^* = N_A^* = g_A^* = 0$, and, from (8.8), $\frac{\partial Y}{\partial K} = \alpha\frac{Y}{K} = \rho + \delta$. ■

References

[1] Aghion, P., and P. Howitt, 1992. A model of growth through creative destruction. Econometrica 60, 323-351.

[2] Aghion, P., and P. Howitt, 1998. Endogenous Growth Theory. MIT Press, Cambridge (Mass.).

[3] Alvarez, M. J., and C. Groth, 2001a. Too little or too much R&D? Mathematical supplement, http://www.econ.ku.dk/okocg/.

[4] Alvarez, M. J., and C. Groth, 2001b. Untitled working paper, Copenhagen.

[5] Barro, R., and X. Sala-i-Martin, 1995. Economic Growth. McGraw-Hill, New York.

[6] Basu, S., 1996. Procyclical productivity: Increasing returns or cyclical utilization? Quarterly Journal of Economics 111, 709-751.

[7] Benassy, J. P., 1998. Is there always too little research in endogenous growth with expanding product variety? European Economic Review 42 (1), 61-69.

[8] Ciccone, A., and K. Matsuyama, 1996. Start-up costs and pecuniary externalities as barriers to economic development. Journal of Development Economics 49, 33-59.

[9] Cooley, T. F., and E. C. Prescott, 1995. "Economic growth and business cycles". In: T. Cooley (ed.), Frontiers of Business Cycle Research. Princeton University Press, NJ, pp. 1-38.

[10] Dalgaard, C-J., and C. T. Kreiner, 2001. Endogenous growth: A knife-edge or the razor's edge? Working paper, Copenhagen.

[11] Ethier, W. J., 1982. National and International Returns to Scale in the Modern Theory of International Trade. American Economic Review 72 (3), 389-405.

[12] Griliches, Z., 1989. Patents: Recent Trends and Puzzles. Brookings Papers on Economic Activity. Microeconomics, 291-319.

[13] Grossman, G. M., and E. Helpman, 1991a. Innovation and Growth in the Global Economy. The M.I.T. Press, Cambridge (Mass.), Chapter 3.

[14] Grossman, G. M., and E. Helpman, 1991b. Innovation and Growth in the Global Economy. The M.I.T. Press, Cambridge (Mass.), Chapter 5.

[15] Howitt, P., 1999. Steady endogenous growth with population and R&D inputs growing. Journal of Political Economy 107 (4), 715-730.

[16] Jones, C. I., 1995. R&D-based models of economic growth. Journal of Political Economy 103 (4), 759-784.

[17] Jones, C. I., and J. C. Williams, 1998. Measuring the social return to R&D. Quarterly Journal of Economics 113, 1119-1135.

[18] Jones, C. I., and J. C. Williams, 2000. Too much of a good thing? The economics of investment in R&D. Journal of Economic Growth 5 (1), 65-85.

[19] Jones, L. E., and R. E. Manuelli, 1990. A convex model of equilibrium growth: Theory and policy implications. Journal of Political Economy 98, 1008-38.

[20] Kortum, S., 1993. Equilibrium R&D and the patent-R&D ratio: US evidence. American Economic Review 83, 450-57.

[21] Norrbin, S. C., 1993. The relationship between price and marginal cost in US industry: a contradiction. Journal of Political Economy 101 (6), 1149-1164.

[22] Romer, P.M., 1990. Endogenous technical change. Journal of Political Economy 98 (Suppl.), 71-102.

[23] Stokey, N. L., 1995. R&D and economic growth. Review of Economic Studies 62, 469-489.

[24] Seierstad, A., and K. Sydsæter, 1987. Optimal Control Theory with Economic Applications. North-Holland, Amsterdam.