The demographic transition and neo-classical models of balanced growth

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Summary
A major concern by scholars in demo-economics is the endogenous explanation of the demographic transition. In this paper, after having reviewed some main results of the recent literature on the subject, we discuss the insight that is provided by Solow-type neo-classical models once they are endowed with "transitional" assumptions. Although Solow's model is usually considered unable to explain the main facts of the transition (Chu 1998), we believe that it may still offer insight on fundamental aspects of modern growth, once we adequately endow it with hypotheses which are coherent with the demographic transition. As a first step we critically discuss the main avenues through which the typical patterns of the demographic transition may be embedded within the model of Solow, by distinguishing the rational perspective, in which the "transition" follows from a maximising framework, from empirical and diffusionist approaches. The ensuing formulations of the transitional mechanism are embedded in simple neo-classical models, that can be used in order to study the main dynamical consequences of the transition on economic growth. The effects of the transition are subsequently analysed under both constant returns to scale (CRS), and decreasing returns to scale (DRS). DRS, which are typical of developing countries, lead to important differences, compared to CRS, which have not (with the exception of Strulik 1999), so far, been stressed in the literature on growth. Our overview and analyses give insight on important aspects such as: i) the onset of poverty traps, and the historically observed mechanisms of escape; ii) a more flexible notion of convergence which appears a powerful explanatory tool for empirically observed dynamics, iii) the need of DRS as unified models of long term historical growth.

1. Introduction

Which are the main economic determinants of the demographic transition (DT)? This question presently represents a major challenge for those scholars in demo-economics aiming at finding endogenous explanations of the transitions between great historical regimes, see for instance Galor and Weil (1999). A generally accepted view is that Solow's (1956) model is hardly able to provide any explanation of the transition. This is expressed, for instance, by Galor and Weil (1999), and Chu (1998,133), who, emphasising the gain permitted by "new growth" contributions, such as the Becker-Murphy-Tamura model (Becker et al. 1990), states "... *the explanation of the demographic transition has not been successful under the neo-classical model of Solow, for it typically predicts a converging steady-state growth rate of per capita income, which is incompatible with the diverging development paths among countries observed over the past 50 years. Moreover, Solow's model is also weak in predicting the relationship between income growth rate and population growth rate. It is well known that in Solows's model the steady state rate of per-capita income is a decreasing function of the population growth rate.*" (Chu 1998, 133).

Though generally accepted, the previous reasoning is, to our mind, exaggerated. The relevant matter can not be whether the original Solows's model with a constant rate of change of the supply of labor can predict the demographic transition, or explain its timing and forms, obviously doesn't. Rather the point is whether, adequately equipped, i.e. by postulating sound "transitional" hypotheses, can the Solow's model offer insight on fundamental aspects of modern growth. An example in this direction is offered by Strulik (1997,1999) who has investigated the consequences of transitional hypotheses in a Solow-type 3-dimensional model with learning by doing, and a Solow-type model for the developing world embedding Keynesian features.

The present paper is subdivided into three main parts. In the first part we review some main aspects of the DT by aiming to combine the distinct perspectives of historical demography and growth theorists. In the second part we critically discuss the main avenues used to embed the transition in economic models, i.e. maximising frameworks (for instance Jones 1999) versus empirical ones, emphasizing the asynchronous patterns of decay of mortality and fertility as per-capita income increases (Strulik 1997) versus "diffusionist" frameworks. This latter approach (for instance Rosero-Bixby and Casterline 1993 and references therein) argues that diffusion effects have often proved to be more important than demand-supply mechanisms in explaining the transition.

Finally, in the third part we tackle our central problem, and try to offer an overview of the effects of transitional assumptions on simple neo-classical models in presence of both constant (CRS), and decreasing (DRS) returns to scale. First we consider a traditional CRS Solow's model. In absence of technical progress, the sole population mechanisms leads to the existence of multiple equilibria, with a stable "poor equilibrium", i.e.a poverty trap, coexisting with a stable rich ("modern") equilibrium. This allows an interesting discussion of the plausible historical mechanisms that could have operated in historical times to provide the escape from the poverty trap. Moreover, the well known "convergence" statement arising in Solow's model is replaced by the more general result that countries that escaped the poverty trap with different endowments of capital would possibly experiment an initial divergent, rather than convergent, phase, which should convert to a convergent regime only in the long (perhaps very long) run. This result thereby provides a powerful explanatory tool for observed dynamics, and an answer to the critical difficulty of Solow's model toward observed facts, as previously quoted from Chu. The introduction of technical progress in the basic model does not offer interesting additional results . This fact seriously questions the use of CRS models in the modelling of historical demo-economic growth. We therefore consider also a neoclassical model with DRS. DRS, which are typical of developing countries, lead to important differences, compared to CRS, which have not, with the exception of Strulik (1999) so far, been stressed in the literature on growth.

The DRS case displays, apart the interesting features already present in the CRS case, more interesting and plausible phenomena. These are, first, the coexistence of stable poor "malthusian stagnation" equilibria with stable states of "modern" long term balanced growth, as historically observed. This fact suggests that DRS models are much better than CRS, as candidates for a unified modelling of "very-long" term economic growth. Moreover the DRS model displays the appearance of persistent endogenous oscillations around the "malthusian stagnation" equilibrium when the population transition is considered jointly with the, quite well documented, saving transition. This latter result, obtained via the most standard growth model for long term growth, offers a nice alternative perspective, i.e. a purely endogenous one, to the traditional exogenous explanations of oscillations around the stagnant malthusian equilibrium.

This paper is organised as follows. The review on the DT is carried out in section two. Section three is devoted to our discussion of the different modelling frameworks of the transition. Section four and five study some CRS and DRS Solow-type models. Section six briefly considers the problem of stably oscillating balanced growth in descriptive neoclassical models. Conclusive remarks follow.

2. The demographic transition and its economic determinants: a review

2.1 The demographer's perspective

With the phrase demographic transition demographers mean the set of dramatic changes by which, starting in Western Europe, particularly England and France, during the XVIII century, demographic systems moved from their Ancien Régime, in which stationarity was maintained via a highly "expensive" balancement between high levels of fertility and mortality, to their modern regime, characterised by low fertility and mortality (Chesnais 1992, Livi-Bacci 1997). The time scales of the demographic transition in the western world have been very long and largely overlapped with the industrial revolution, so that simple explanations are not easy. Following the point of view of historical demography (Livi-Bacci 1997,1998) the essential mechanisms are the following: 1) at some stage the achievement of a threshold in the level of technology (or an increase in its pace of growth), allowed an increase in the agricultural surplus, which in turn lead to three sets of macro effects: a) a sharp reduction in mortality; b)an increasing trend in per-capita income and wellbeing, c) a fall in the demand for labour in agriculture which was the condition for a transformation of the social organisation, from rural to urban. The slowly increasing historical dynamics of technology appears therefore to have been a main engine of the process. The ensuing reduction in mortality appears to be mainly the consequence of endogenous factors[1], e.g. increases in productivity and well-being, which reduced food crises, improved the human diet, and allowed a dramatic decrease in major killers such as typhus and TB. 2) the reduction in mortality is the engine favouring the subsequent reduction in fertility[2], via the great shift from quantity to quality of children (there are are many economic factors intervening in this complex process, see Galor and Weil (2000)). From the demographic standpoint the chain is, roughly, the following: the reduction in mortality increases the reproductivity of the population, leading initially to larger families. But in urban areas children have larger costs and become income producers much later compared to rural areas, so that fertility decline became a need. The diffusion of birth control (much more than the Malthusian postponement of marriage) was the main tool through which fertility decline became possible (and easier in urban areas where the social control by religion became less and less important). 3) the output of the process has been a transient period of fast population growth, mainly due to the asynchronous decay of mortality and fertility, and per-capita income. Population growth eventually came to an end, due to the continued decrease in fertility, whereas growth in per-capita income has not been interrupted.

Demographic approaches are mainly descriptive, focusing on heterogeneities in timing at onset, pace, and local features of the transition, and no attempts has been made to develop simple formal demo-economic explanations of the transition. Major questions, such as on the inevitability of the transition, have not been posed at all. Very recently the transition has become the object of renewed interest by (demo-) economic growth (Galor and Weil 2000 and refs therein, Jones, 1999 and refs therein).

2.2 A unified frame for the definitions used in the literature

In order to manage what is nowadays a jungle of papers with a huge number of different definitions, we introduce some fundamental definitions in the endogenous growth theory with population.

Poverty trap: traditionally (Barro and Sala y Martin) it is a stable steady state whose "name" can be so motivated: 1) "poverty" because it has low levels of per capita output and capital stock, 2) "trap" because if agents attempt to break out of it then the economy always tends to return to it. But the meaning of "trap" is more clear if it co-exists with (at least) another equilibrium with better properties in terms of "welfare" which is either i) repelling (so that when economy could – due to some shock – approach it, the trap would inexorably re-swallow it), or ii) attracting but with a basin

[1] Exogenous factors (disappearance of plague etc) as well played a role but probably a minor one.

[2] In some cases however there is also evidence (Dyson and Murphy, 1985) that fertility temporarily rose, thanks to the increased well being, before starting to decline.

of attraction so far from the "poverty trap " region as to be unattainable for the economic variables. The search for models with endogenous mechanims of escape from the poverty trap has been a main "target" for growth theorists[3]. It is easy to see that, by only focusing on the dynamics of per capita physical capital as in the solowian model, in order to attain the "target" (for instance in order to have the interval of rising average product, necessary to have multiple equilibria) there are few possinilities: i) to assume increasing returns to scale for all the factors through learning by doing or other externalities, 2) to assume a non-linear saving function, 3) to suitably endogenise population growth. While the first two candidates have been largely used, and in particular the first perhaps is, directly or indirectly, the "engine growth" of all the endogenous growth theory, the third, however largely explored, has not been successful, at our knowledge, in the simple framework of Solow 1956. In the present work we attempt to fill this lack.

According to Galor-Weil (1999) three distinct regimes have characterised the process of economic development: the "Malthusian" Regime, the "Post-Malthusian" Regime, and the "Modern Growth" Regime. To fully understand these definitions it is useful to distinguish the macroeconomic point of view from the demo-economic point of view: the first focuses on the behavior of income per capita and technological progress, while the second on the relationship between the level of income per capita and the growth rate of population.

Malthusian Regime: 1) low technological progress and population growth, at least relatively to modern standards, and income per capita roughly constant 2) **positive** relationship between income per capita and population growth.

Post Malthusian Regime: 1) growing income per capita during this period, although not as rapidly as it would during the Modern Growth regime 2) **positive** relationship between income per capita and population growth (still as in the Malthusian Regime). Notice that it is between the other two here described, shared one characteristic with each of them.

Modern Growth Regime: 1) steady growth in both income per capita and the level of technology 2) **negative** relationship between the level of output and the growth rate of population.

Notice that from the demoeconomic point of view the DT is represented from the transition from the Maltusian to the Modern Growth Regime. As we are mainly interested to the relation between economic growth and DT, we consider the two latter regimes. The PostMalthusian Regime would be obviously interesting if the focus were on the Industrial Revolution, the technological progress and other macroeconomic points. Notice that from for the demoeconomic perspective, the Modern Regime is sometimes defined also as *Post-Classical, Anti-Malthusian, Beckerian*.

2.3. The literature about transition from malthusian to modern growth regime: demographic transition is represented?

Research on the interaction between income growth and fertility is not new in economics (i.e. Razin-BenZion, 1975) but only more recently the literature has focused on the existence of different long-run regimes corresponding to the historical transition from the ancient to the modern world. Basically all this literature represents the world as characterised by multiplicity of equilibria, one equilibrium with a low level of output and a high rate of fertility, while another with a high level of output and a low rate of fertility. Starting from the research agenda of Becker (1988) which explicitly aims to model economies which have at least two stable equilibria which can be 'ranked' with respect to some definition of well-being, the one («good») with high fertility (population) and low levels of capital, the other («bad) with low fertility and high levels (or growth) of capital, Becker et al. (1990), obtained multiple equilibria: their model takes explicitly account for the

[3] The same terminology may express a different concept in other authors: i.e Kögel-Prskawetz (2000, p. 2) claim that: " . . economy will be trapped in a situation where no sustained growth of per capita income can occur. This trap is commonly labeled Malthusian trap" and "we label the simultaneous take-off in economic growth and population growth as escape from the Malthusian trap". In our opinion these definitions are restrictive: they reflect the emphasis on the "forever sustained" exponential growth which pervades all the endogenous growth theory.

human capital of parents and (differently from the previous models) for its influence on i) the cost (in terms of time) of childrearing, ii) on the market wages. Their crucial assumption is that returns on the human capital are increasing rather than decreasing function of the stock of human capital. According to the initial conditions ("history and luck") the model shows how it is possible either to stay "trapped" in a low per capita income equilibrium and high fertility or to approach another equilibrium with high per capita income and low fertility, or, possibly, within an opportune parametric configuration to approach even a self-sustained growth path for the per capita human capital. Whether these equilibria depict Malthusian or Modern (Beckerian) states seems to be indefinite: the authors show that "*fertility and the steady state rate of growth in per capita incomes could be either negatively or positively related among countries, or over time in a given country, depending on why growth rates differed*" (p.S24).We notice that as fertility tends to be high when the human capital is low because the childrearing cost is cheaper, then we are not sure that when the human capital exponentially grows the fertility remains positively bounded (or furthermore that its equilibrium value exists). Lucas (1998) aims to explain differences in population dynamics between the pre Industrial revolution and the Modern era, building a model of the economy, in which endogenous fertility and human capital accumulation are the main features; he shows that the economy can exhibit two different steady states, Malthusian or Modern, depending on the value of a certain parameter governing the private return to human capital accumulation (similar to the Becker et al.'s results).

While the role that population plays in these latter models is heavily dependent on the assumptions about the so-called "parental altruism", in the early years other models based on the simple structure of the Solow's model investigate the effect of the population endogeneization without a microfoundation of the fertility decision (Cigno 1981, 1984a, 1984b)[4]. On the other part Yip-Zhang (1997) depart from the model of Benhabib-Perli (1994) to test if a similar model with endogenous fertility may reproduce the same result about the emergence of indeterminate equilibria. The fertility function is microfounded following the existing literature by allowing the number of children directly to enter the utility function of the representative household. Then it is shown that it can exist either a unique equilibrium globally determined or multiple "indeterminate" equilibria: in this latter case two equilibria emerge, the one Malthusian and the other anti-Malthusian (or Modern). Also these two latter models belongs to the class of models with multiple equilibria (and two possible asymptotic regimes, reflecting respectively the Malthusian world or the Modern Western World), which contains Becker et al. (1990), Palivos(1995), Tamura (1996), Lucas (1998), Strulik (1997,1999).

A second strand of literature deals explicitly with the relation between economic growth and the Demographic Transition. In this literature very different explanations of the demographic transition are provided, and also when the demographic transition is assumed but unexplained, very different effects of this latter are discussed, as the following brief survey attempts to illustrate. Moreover very recently some papers attempt to unify the explanation of the Industrial Revolution and the DT. Azariadis- Drazen (1991) argue that the DT may be caused by an increase in the bargaining power of children as production becomes more urban oriented. Ehrlich-Lui (1991) explain the DT as the result of the interaction between "companionship" and "altruism". Dahan-Tsiddon (1998) argue that the "*it is not income per se, but it is the return to education….that brings income and fertility together*" (p.47) and in presence of an unequal income distribution the DT occurs when both the population of the poor begins to invest in human capital and the net incentive to invest in human capital is sufficiently high. The approach originating from Cigno is used more recently by Strulik (1997) to investigate the joint consequences of a learning-by-doing effect (as stimulated by the empirical work of Boserup, 1981) on the production side, with a behaviour

[4] In these papers the population is endogenised by supposing that the RPG is positively related to per capita consumption (which is of course, as in Solow, simply proportional to income) and inversely related to the per capita capital. While the first term is usually Malthusian, the second term would be a proxy of the degree of industrialization, which, with its concomitant urbanization, would have a depressive effect on birth rates.

mimicking the Demographic Transition on the population side. The joint effects of learning-by doing and DT allow the author to prove the existence and stability of a low-income equilibrium and a high-income equilibrium in a neoclassical growth model à la Solow.

In a some sense our model has as "benchmarks" with which to make comparisons the models of Becker et al.(1990) and Lucas (1998) (and more in general all the models generating multiple equilibria). Although these two models, apart from the endegeneisation of the fertility, are at all different from ours in that they are i) microfounded according to a different story, and 2) are centred on the role of human capital accumulation, they share with our model the focus on the population dynamics in a Malthusian and in a Modern era, and in particular the following same interpretation: the Malthusian and the Modern eras are different steady states of the same model.

Another more recent class of models aims to explain the entire transition from ancient ("Malthusian") world to the Modern era for what concerns the occurrence both of the Industrial Revolution and the DT. Among these latter, some are able only to explain the Industrial Revolution (Hansen-Prescott, 1999; Jones, 1999; Kögel-Prskawetz, 2000); other models also explain, in addition to Industrial Revolution, the Demographic Transition (Galor-Weil, 1999; Jones, 1999). It is worth to notice the main feature that distinguishes the models with multiple equilibria (as Becker et al., 1990 and Lucas, 1998) from the class of models (Hansen-Prescott, 1999) that attempt to provide a unified theory that can account for the facts both of the Malthusian era and of the Modern era, explaining also the inevitability of the transition between the two eras (the so-called industrial revolution): in the latter models remarkably this transition occurs without no changes in the structure of the economy (that is parameters describing preferences, technology and policy), as on the contrary it must occur in the previous models to trigger the transition between the two equilibria (from Malthusian towards Modern), and moreover the equilibrium implied by this theory is unique instead of multiple. The Hansen-Prescott's model is a standard neoclassical growth model à la Solow with one good, but with two available technology, the first, defined as Malthusian technology, (three inputs, capital, labour and land), the second defined as Solowian technology (two inputs, capital and labour), both with constant return to scale to the production: in the early stages of development only the first technology is used, but eventually technological progress makes the Solowian technology profitable, and both technologies are employed; in the limit only Solowian technology is used and the model collapses in the standard neoclassical Solowian model. This latter model is however silent as to why population growth increase with the per capita income before the industrial revolution and decrease with per capita income after such a revolution. Close to paper of Hansen-Prescott is Kògel-Prskawetz (2000), where it is assumed that the two technologies (Malthusian and Solowian) produce two different consumption goods rather than the same good as in Hansen and Prescott. Moreover they endogeneise fertility which results, from a standard utility maximization, increasing with the per capita income. However they explain the transition from Malthusian to PostMalthusian Regime, are unable to explain the DT and the passage to the Modern Regime.

Contrary to Hansen-Prescott's model, a theory which attempts to be consistent with the long transition process in which not only the technology and output growth, but also the evolution of population and in particular of the fertility behaviour, move across the different eras described above, is developed by Galor-Weil (1996,1998) and Jones (1999). In Galor –Weil (1998) the main elements of the model are the following: i) technological progress raises the rate of return to human capital and hence induces parents to substitute quality for quantity of children (perhaps reflecting the strong rise in schooling and the hint of the demographic transition in Europe over the course of the 19th century); ii) the choice of parents regarding the education level of their children, through the positive effects that children with high levels of human capital may have on the inventions, influences the speed of technical progress; iii) the size of the population positively influences the growth rate of technology; iv) the increase in population implies that the land to population ratio falls, and the wage falls. With this latter classical assumption, static technology implies that the size of the population is self-equilibrating. A stable Malthusian steady state prevails over long periods of time, with stationary per capita income and population. Moreover the static technical progress also implies that the return to human capital is low, preventing any incentive for parents to substitute child quality for quantity. However, if a sufficiently fast technological progress relaxes the land constraint, then wages may rise. The “trigger” in order to causes the Malthusian steady state to vanish in the long run is the effect of population size on the rate of technical progress, which in turn will raise the return to human capital sufficiently so as to induce parents to provide their children with some human capital. At this point, a virtuous circle develops: higher human capital raises technological progress, which in turn raises the value of human capital. But the transition in the population growth behaviour how and why can be “triggered”? And why, when the above economic virtuous cycle is already began, the population, over a long time, continues to behave as in the Maltusian case, as it effectively makes in the regime called - by these authors - Post-malthusian? Their answer is that, since the increased technical progress initially has two effects on population growth: 1) it eases households' budget constraints, allowing them to spend more resources on raising children; 2) it induces a reallocation of these increased resources toward child quality, then when the former effect dominates, and so population growth rises along with per capita income growth, the Post-Malthusian regime is established. But the second effect, via higher human capital, can further increase the speed of the technical progress and eventually triggers a demographic transition: the return to child quality continue to rise, the shift away from child quantity becomes more significant, population growth declines, and output growth rises.

Our major criticism to this view is that the true “engine” of this complicated transition should be the postulated effect of population size on the rate of technical progress, which is at least dubious (as in all the so-called “idea-based” models).

But if this latter paper makes a praiseworthy attempt to encompass in an unified model the transition between all the three distinct regimes that characterised the history of the development, other papers are content with an explanation of the transition between only two regimes. Among these latter, Kremer (1993) provided the first attempt at modelling this long transition, generating a transition from the proximity of a Malthusian equilibrium to the Post-Malthusian regime.

Also his model has crucially based upon a simple relationship between technology and population size and simply assumes that population growth increases with income at low levels of income and then decreases with income at high levels of income.[5] The major criticisms are that still is assumed – as crucial engine – “more heads, more technical progress”, and that, mostly, this latter assumption does not lead, however, to the demographic transition. Galor and Weil (1996) develop a model that is consistent, unlike Kremer’s one, with the transition between the Post-Malthusian Regime and the Modern. A crucial feature here is the introduction of a fixed factor of production. There are elements to the model: 1) wage differential between men and women is decreasing with the economic growth; 2) capital accumulation or technical progress raise women's relative wages, since capital is more complementary to women's labour input than to men's, 3) consistently with empirical evidence, increasing women's relative wages reduces fertility by raising the cost of children more than household income, whereas an increase in male’s wages increases fertility due to the pure income effect; 4) lower fertility raises the level of capital per worker; 5) there exist two sectors, the one modern, the other “traditional” (presumably work at “home”); in this latter there is a production technology for producing market goods in which time spent producing can be partly used for child rearing and with which the women’s marginal product is unaffected by technological progress and capital accumulation. At the beginning the economy is close to a Malthusian equilibrium. The crucial assumption in order to start the development is still that the technical progress is driven by the size of population, so that, as population rises and technical progress is

[5] In order to microeconomically justify the assumed relation, he argues that this pattern could follow if raising children entails costs both in goods and time, mortality fall with income and the utility function requires lower bounds on consumption level and the number of children.

sufficiently rapid, the land to population ratio falls, but wages may increase nevertheless, allowing for further technical progress and capital accumulation. When capital and technology rise, family income increases via man's wages, while women wages in the home sector do not change. Fertility increases due to the pure income effect. Then the economy shows the feature of a Post-Malthusian Regime. However, as the modern sector becomes sufficiently productive, women join the labour force and fertility declines due to the substitution effect and the economy experiences a demographic transition.

The story told in this latter paper in order to explain the transition is based, in contrast with Galor-Weil (1998), on the approach of Mincer (1963) rather than Becker's one. This two papers, then, are a useful example of a modern reappraisal of the two main neoclassical theoretical explications of the observed decline of the fertility associated with the economic growth: namely, on one side, according to Mincer, the raise of the relative price of children - due mainly to an increased opportunity cost for women for childrearing - which has overcame the positive effect due to an increased income; on the other side, according to Becker, the substitution of the child quality for quantity. Apparently both the stories are adequate to explain the transition.

Finally Jones (1999) builds a model capable not only with its analytical results but also with a well calibrated simulation to reproduce the industrial revolution as well as the DT. The story told is something like the following: people was successful, however by efforting for very long time, in producing a single idea, but once this single invention is occurred consumption and fertility rose, implying an increase in the rate of population growth; this latter increase has made more people available to find new ideas, so that successive new ideas was occurred more quickly. Due to an aggregate production function characterised by increasing returns to scale to accumulable factors, the positive link "new ideas- population" leads to accelerated rate of population growth and consumption growth. Furthermore two very remarkable facts are that: 1) the simulated economy behaves as the DT predicts; 2) in the very long-run the population can become, however gradually, stationary while the consumption grows exponentially.

To sum up it is worth to remark that although there are many recent papers focusing on the transition process in which not only the technology and output growth, but also the population behaviour evolves, only a few of those are capable to represent the DT process or at least to assume the DT as a "fundamental stylised fact" with which to reckon.

3. Representations of the DT in growth models: optimising versus diffusionist perspectives

The aim of the present section is that of discussing some main avenues through which the DT may be modelled in descriptive growth models, such as the Solow's one. As we will see, in the final analysis, all these approaches lead to endogenise the supply of labour via a humped function of per-capita income, that will be used for subsequent analyses.

3.1. The transition as the outcome of an optimal choice

Here we enrich Jones' (1999) formulation (not considering saving) by adding an exogenous non zero saving rate, in order to make Jones' formulation compatible with the accumulation side in Solow's model. Each individual has, in each period of time, an endowment of one unit of labour which can be used to obtain consumption or children. Let h= fraction of time spent working ($1-h$=time spent producing children), w=earnings per unit of time worked, s= constant saving rate, $c°$=the (constant) subsistence consumption, $b°$=the constant number of children independent from the time spent on childrearing. The utility of individuals depends on consumption (c) and number of children (b), and has the form (Jones 1999):

$$u(c,b) = \frac{(1-m)(c-c^{\circ})^{1-g}}{1-g} + \frac{m(b-b^{\circ})^{1-h}}{1-h} \qquad 0 < m < 1 \ ; \ 0 < g, h \le 1 \qquad (3.1.1)$$

The individuals take w as given and solve the problem: $\max_{c,b,h} \; u(c-c°, b-b°)$ subject to the restraints: i) $c=(1-s)wh$; and ii) $b=f(1-h)$. This latter says that each unit of time spent producing children produces f births with $f>b°$. A simple riformulation of the constraints i) and ii) obtains the usual budget constraint $c+\frac{b(1-s)w}{f}=(1-s)w$ where *(1-s)w* is the disposable income and *1/f* is the (per child) cost of childrearing. Let $\hat{c}=c-c°; \hat{b}=b-b°$. It is easy to show that the "net" optimal quantity of children and the optimal consumption respectively are

$$\hat{b}=\left[\frac{fm\hat{c}^{g}}{(1-s)w(1-m)}\right]^{\frac{1}{h}} \quad ; \quad \hat{c}=(1-s)w(1-\frac{b}{f}) \qquad (3.1.2a,b)$$

Remembering that $\hat{c}=c-c°$; $\hat{b}=b-b°$, one can write (3.1.2a) as

$$b-b°-\left\{\frac{mf}{(1-m)(1-s)w}\left[w(1-\frac{b}{f})-c°\right]^{g}\right\}^{\frac{1}{h}}=0 \qquad (3.1.3)$$

which defines the relation between the optimal fertility rate and the wage in implicit terms. By the implicit function theorem it is possible to show (Jones 1999) that (3.1.3) actually defines b as a humped function of w, therefore mirroring a major stylised fact of the demographic transition.[6] For sake of simplicity, we consider here the special case $\gamma=\eta$, for which the relation (3.1.3) may be put in explicit form:

$$b=\frac{b°f+Hf\left((1-s)w^{\frac{h-1}{h}}-c°w^{-\frac{1}{h}}\right)}{\boldsymbol{W}} \quad ; \quad H=\left[\frac{fm}{(1-s)(1-m)}\right]^{\frac{1}{h}} \quad ; \quad \boldsymbol{W}=f+H(1-s)w^{\frac{h-1}{h}} \qquad (3.1.4)$$

Notice that this special utility function shows a high elasticity of substitution between consumption and children (e): indeed e=1/η>1 and the elasticity is decreasing with η (at the limit when η tends to one, the elasticity tends to become unitary as in the Cobb-Douglas case). A differentiation shows that

$$\frac{\partial b}{\partial w}=\frac{Hf\left[c°-w(1-\boldsymbol{h})(1-s)\right]}{\boldsymbol{h}w\Omega}+\frac{Hf\left[c°-w(1-s)\right](1-s)(1-\boldsymbol{h})w^{-\frac{2}{h}}}{\boldsymbol{h}\Omega^{2}}+\frac{Hfb°(1-s)(1-\boldsymbol{h})w^{-\frac{1}{h}}}{\boldsymbol{h}\Omega^{2}}$$

The latter expression suggests that, coeteris paribus, a high subsistence consumption, a high propensity to saving, and a high "natural" fertility imply" imply that the hump in fertility occurs for larger values of the wage. This implies that these factors could have played a role in temporally

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6 Jones (1999) remarks*: "The traditional income and substitution effects are reflected in the second term. As the wage goes up, the income effect leads individuals to increase both consumption and fertility. The substitution effect, on the other hand, leads people to substitute toward consumption and away from fertility: the discovery of new ideas raises the productivity of labour at producing consumption, but the technology for producing children is unchanged. If g<1, then the substitution effect dominates, while if g>1,the income effect dominates. As usual, if g =1,i.e.with log utility, these two effects offset. A third effect not traditionally present is reflected in the first term: as the wage rises, the subsistence consumption level which the consumer i required to purchase gets cheaper, leading consumers to have more after-subsistence income to spend on both more children and more consumption. This effect disappears as the wage gets large. The assumption that 0 <g<1,then, leads the subsistence effect to dominate for small values of the wage and the substitution effect to dominate for large values of the wage, producing one component of the demographic transition: fertility rises and then falls as the wage rate rises*."(p. 9).

delaying the onset of the population pulge (i.e. the attainment of the moment of maximal population growth).

The previous development can be used to endogenise the population growth in descriptive growth models, such as the Solow's (1956) model, in which the supply of labour L is assumed to be exogenously growing. In general terms the quantity of labour supplied for production is related to the total population in the work age span (N) by $L = hN$ where h is the participation rate. By assuming that h is constant over time, it then simply holds $\hat{L} = \hat{N}$. Let us assume that (with some further simplifying assumptions) that $\hat{N} = b(w) - m(w) = n(w)$, where b and m respectively denote the birth and the death rates in the population. Obviously, even in the simplest case, i.e. m is taken as constant, the one-humped shape of b implies that n is one-humped as well. If n is also non-negative, which can be obtained by simply taking $m=b_0$, then $n(w)$ faitfully mirrors the demographic transition.

The present approach offers a nice route to an endogenous modelling of fertility transition within the descriptive growth models, which is in full closed form, and therefore potentially fruitful of interesting results.

3.2 Empirical approaches

A highly stylised "model" of the DT typically used by demographers represents the transition as an historical phase of population growth, which separates the ancient and modern demographic eras, mainly characterised by stationarity. This phase of growth is the outcome of the observed asynchronous decline, usually s-shaped, of mortality (falling first) and fertility (Chesnais 1992) over time. This empirically-based argument is expanded in Strulik (1997,1999) who models the population birth (b) and death (m) rates during the transitional regime by means of two asynchronously decreasing logistic-alike functions of per-capita income y.[7] As a consequence the corresponding rate of change of the population, i.e. the difference $n(y)=b(y)-m(y)$ turns out to be a humped function of per-capita income (see Remark 1,part a)). Such a function proves to fit very well classical "transitional" data sets (Strulik 1997,1999). A theoretical justification based on a simplified overlapping generation argument is also given in Strulik (1999).

The dynamical consequences of a humped $n(y)$ function for macro-economic growth models are far from being trivial. As Strulik (1997) correctly noticed the traditional demographic approaches to the DT are basically a descriptive comment of the peculiar time patterns of the components of population growth observed during the transitional regime. This may lead to the wrong impression that the transition has simply been "inevitable" as a consequence of the historically observed increase in per-capita income. As showed in the next section even the simplest growth model, the neo-classical Solow 1956 model, suggests that this does not need to be the case.

3.2 A model for time diffusion of birth control through spread of information

Though Strulik's choice appears mainly motivated by empirical considerations, we believe there is an important theoretical argument in favour of the logistic-type curves approach, namely the diffusionist argument. As well recognised (Livi-Bacci 1997,1998) there is a strong body of evidence suggesting that the main tool by which fertility decline was made possible during the DT was the diffusion of the practice of birth control. As pointed out in the introduction, the diffusionist paradigm is often taken as the basis for alternative explanations to those based on economic factors, which often provide better fits of observed patterns and pace of the transition (Rosero-Bixby and Casterline 1993, and references therein). More general explanations seek to embody both types of

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[7] His argument follows the traditional lines already discussed in section two: "starting with low income and mortality as well as fertility at very high levels, rising income causes sharp declines in mortality through e.g. better nutrition and health care. With fertility remaining high, rapid population growth occurs. With further rising income per capita fertility declines too, so that in the last phase of the transition a low death rate is balanced by a low fertility rate, and population growth is slow (1997, 289).

effects (Retherford and Palmore, 1983). Here we consider a model for the diffusion of birth control which justifies under very wide conditions the s-shaped time pattern for a typical component of population growth, e.g. the birth rate, during the transition, and then discuss how this may lead to the empirical relations estimated by Strulik. Consider a population experiencing the DT which is composed by two groups, those who are still practicing natural (uncontrolled) fertility, and those who are practising birth control. We assume that the transitional decrease in mortality has already occurred, and that the "natural" group has high fertility and would, if alone, grow exponentially, whereas the "controlled" group has a lower fertility. Let $X=X(t)$ and $Y=Y(t)$ respectively denote the size of the natural and controlled fertility subgroups. Individuals are assumed to move from the natural to the controlled group due to the diffusion of information on birth control, which occurs via both *external* (e.g. the action of the media) and *internal* (inter-human contacts) diffusion.[8] Moreover child are assumed to inherit the fertility attitudes of their parents at birth. The model is the following

$$\begin{aligned} \dot{X} &= b_N X - mX - \left(\boldsymbol{a}X + \boldsymbol{b}\frac{XY}{N}\right) \\ \dot{Y} &= b_C Y - mY + \boldsymbol{a}X + \boldsymbol{b}\frac{XY}{N} \end{aligned} \qquad (3.2.1)$$

where $\boldsymbol{m}$ is the mortality rate, b_X, b_Y denote the fertility rates of both natural and controlled groups, with $b_X > m$, b_Y³m, $b_X > b_Y$. Moreover α is the rate of transition from natural to controlled due to external information, whereas β is the contact rate in internal diffusion. Model (3.2.1) is more general of other models used for the diffusion of birth control in that it is not limited to the situation of a stationary population as in Rosero-Bixby and Casterline (1993). Write the model as

$$\dot{X} = r_X X - \boldsymbol{b}\frac{XY}{N} \quad ; \quad \dot{Y} = r_Y Y + \boldsymbol{a}X + \boldsymbol{b}\frac{XY}{N} \qquad (3.2.2)$$

where $r_X = b_X - \boldsymbol{m} - \boldsymbol{a}$; ; $r_Y = b_Y - \boldsymbol{m}$. Notice that (3.2.2) shows that the $\boldsymbol{a}$ parameter simply act as a component of the "intrinsic" growth rate of the X population. Therefore if $\boldsymbol{a}$ is so large that r_X£r_Y the problem becomes trivial. We therefore assume that r_X³r_Y. The model (3.2.1) is homogeneous, and therefore amenable to analytical solution. Let us consider as new variables the total population $N=X+Y$, and the controlled fraction $\boldsymbol{j}=Y/N$. One gets the decoupled equations

$$\boldsymbol{j} = (1-\boldsymbol{j})(\boldsymbol{a} + (\boldsymbol{b} - (b_X - b_Y))\boldsymbol{j}) \quad (3.2.3)$$

and:

$$\frac{\dot{N}}{N} = r_Y + (b_X - b_Y)(1-\boldsymbol{j}) \qquad (3.2.4)$$

where $\boldsymbol{j}$ belongs to [0,1]. Notice that the initial condition $\varphi(0)>0$ is not necessary due to the existence of external diffusion (we could also posit $\varphi(0)=0$). Equations (3.2.3)-(3.2.4) can easily be solved analytically but we are not interested in doing so now. Notice rather that there are two qualitatively interesting cases. First, if $\beta-(b_X-b_Y)>0$, i.e. $\beta>b_X-b_Y$, then (3.2.3) only has the nontrivial equilibrium $\boldsymbol{j}_1=1$ which is globally asymptotically stable (GAS). In other words, when the effects of inter-human communication are sufficiently strong, then in the long term all the population choose, in relative terms, to move to birth control. This means that the higher fertility of the natural group is made uneffective by the 2migration" toward the lower fertility group. If $r_Y=0$ the total population becomes stationary in the long term, whereas it continues to grow if $r_Y>0$. Second, if $\boldsymbol{b}-(b_X-b_Y)>0$, i.e. $\boldsymbol{b}<b_X-b_Y$, (3.23) also has the equilibrium $\boldsymbol{j}_2=\boldsymbol{a}/(b_X-b_Y)-\boldsymbol{b}$ which is demo-

[8] Rosero-Bixby and Casterline (1993) suggests three main types of effects which may lead people to switch to a different group: not only information flows, but also demonstration effects, based on the experiece of other people which evidence the benefit of the transition, and changes in normative contexts.

economically meaningful for $\beta+\alpha<b_X-b_Y$, i.e. when the overall rate of information diffusion is below a prescribed threshold. In this case φ_2 is GAS implying that in the long term there is a coexistence between the two groups, and the total population will continue to grow unless the fertility of the controlled group is sufficiently below replacement.
Consider now the overall birth rate in our population, which is a weighted average of the birth rates of the natural and controlled fertility subgroups:

$$b_N(t) = b_X(1-\varphi) + b_Y\varphi$$

Notice that in both cases above considered, the time behaviour of $\varphi(t)$ will that of an s-shaped function, i.e. very much alike logistic functions. This implies that in particular under the first case, and always postulating $\varphi(0)=0$, the overall fertility rate b_N will decline logistically over time from its natural fertility level b_X to its controlled level b_Y.

The previous analysis shows that, as a consequence of the diffusion of birth control, the time profile of the birth rate is that of an s-shaped decreasing function, as largely documented by the empirical evidence on the DT. In order to derive the s-shaped relations with per-capita income (rather than time) used by Strulik (1997,1999) one should consider more general models. We just suggest here that since per-capita income has showed a monotonically increasing, therefore invertible, historical time trend $y=T(t)$, during the transitional era, then the s–shaped time profile of the birth rate $b_N(t)$ is usually maintained when one considers the function $b_N(T^{-1}(y))$ (and of course it is preserved in empirical analyses of the same relation).

4. The demographic transition and the Solow's model: the case of constant returns to scale

4.1 A Solow-type model with DT

In this section we start to investigate the dynamical consequences of the DT, within the descriptive neoclassical model. We start from the Solow 1956 model (without technical progress). According to the discussion of section 3, the DT may be, in gross terms, represented by replacing the exogenous rate of change of the supply of labour with a positive and humped function of per-capita income y. Denote such function as $n(y)$. We therefore obtain the following "transitional" Solow-type model

$$\dot{k} = s(y)f(k) - (\delta + n(y))k \qquad (4.1.1)$$

where as usual k=the capital labour ratio, $f(k)$=a standard production function (in per capita terms) with CRS satisfying Inada's condition, s=the saving rate (assumed to be income dependent for generality), δ=the rate of capital depreciation, $n(.)$=the aforementioned rate of change of the supply of labour. Since $y=f(k)$ under CRS, we can write (by writing $s(k),n(k)$ as shotcuts for $s(f(k)),n(f(k))$:

$$\dot{k} = s(k)f(k) - (\delta + n(k))k \qquad (4.1.2)$$

Clearly n has a humped relation with per-capita capital as well. In particular we assume that n starts from zero (or a slight positive value at the beginning of the transition), increases up to a maximum, and finally goes down up to a small positive (or zero) asymptotic value at the end of the DT.[9].

Remark 2. Formulation (4.1.2) is well acknowledged in the literature on economic growth with endogenous population. Already Solow (1956) considered it, and concluded that the endogenizaton of population had no relevant effects on his main results (see also Nelson (1956), Niehans (1963),

[9] We will not be concerned with post-transitional (or "second" demographic transition) phenomena (of which Italy and Spain in the past 15 years are the major example), by which the rate of change of the population could even become strongly negative.

Nerlove and Raut 1998). Subsequently it was observed that such endogenization could produce multiple equilibria. However, all the examples reported up to now, though often based on complicate relationships (Nerlove-Raut 1998) did not produce interesting results: at most a second, but unstable, equilibrium appears. This uninteresting outcome has lead to scepticism about the approach: *"it is clear that merely endogeneizing population growth at the macro-level does not shed light on the shape of n and thus on the nature of the dynamics; a utility-maximising model should be used to elucidate the nature of the function*" (Nerlove-Raut 1998, 1127). Though Nerlove and Raut's observation is in principle correct, we believe that an indisputable exception is represented exactly by the humped "demographic transition" hypothesis which not only is supported by a massive amount of empirical evidence, but moreover, as discussed in the previous section, can be fully micro-funded (see also Strulik (1999), Momota and Futagami (2000)).

4.2 Results

The analysis of (4.1.2) is straightforwardly carried out by usual graphical tools (Barro and Sala-i-Martin, 1995, ch. 2) for 1-dimensional growth models. Let us consider first (fig. 1) the case of a constant saving rate. The function $m_1(k)=sf(k)/k$ has the traditional decreasing form whereas the function $m_2=\delta+n(k)$ mirrors the humped form of the rate of change of the population. The vertical distance $m_1(k)- m_2(k)$ is the rate of growth of k over time, denoted as γ_k. Fig.1 reveals that multiple equilibria may exist. The behaviour of (4.1.2) is summarised by the following

Proposition. Apart the zero equilibrium, system (4.1.2) admits one or three non-trivial equilibria. In the former case let k_1 be the unique positive equilibrium. Then k_1 is always globally stable. In the latter case let the three equilibria be $k_{low}<k_{mid}<k_{high}$. It is easy to show that k_{med} is unstable, whereas k_{low} and k_{high} are locally stable, with respective basins of attraction: $Bas(k_{low}) = (0,k_{med})$, $Bas(k_{large}) = (k_{med},+\infty)$.

The proof of the proposition straightforwardly follows from the inspection of the sign of the rate of change γ_k (the arrows on the horizontal axis in fig. 1 denote the direction of motion of k). The previous proposition leads to the following economic results:

Existence of a poverty trap[10]

In the case of multiple equilibria in (4.1.2), there is a ranking of equilibria in terms of all the per-capita variables, so that the locally stable "low" equilibrium k_{low} is a poverty trap (a "malthusian trap" in the language of demo-economists). Barro and Sala y Martin (1995) argue that poverty traps typically arise as the result of the coexistence of regions of decreasing returns with regions of increasing returns. In (4.1.2) the poverty trap rather arises as the consequence of the (indisputable) shape of the rate of growth of the population during the transitional regime. There is a second interesting case, occurring in the case of the unique equilibrium, namely when the m_1 curve is so low that the existence of k_{mid},k_{high} is prevented. In this case k_1 is a "very poor" equilibrium (coeteris paribus $k_1<k_{low}$), and is stable: therefore k_1 may be termed a poverty trap. In this case the unique equilibrium well represents the case of the "ancient regime" malthusian stagnation (Galor and Weil 1999) during which accumulation was so low that the existence itself of a richer regime was impossible. Since now on we also define the poverty trap k_{low} as the "malthusian" equilibrium, and the rich equilibrium k_{high} as the "modern", or post transitional, regime. Notice that there is an opposite case, when the m_1 curve is so high that the existence of the poor equilibria k_{low},k_{med} is prevented. In this case k_{high} becomes a "virtuous" equilibrium.

Remark 3 (non ineluctability of modernisation) The existence of a poverty trap suggests that the traditional view, which looks at *modernisation* (the industrial revolution versus the DT) as an ineluctable process, is largely uncorrect.

10 See definition is section 2.2.

Once a poverty trap exists, a main problem is of course how to break out of it. Consider, just to fix the ideas a Cobb-Douglas technology $Y = QL(t)^{b} K(t)^{1-b}$. We assume that at some initial stage the economy is emprisoned into the stagnant "one equilibrium" regime (curve m_1^* in fig. 1). For models as (4.1.2) the escape from the trap may only occur as a consequence of policies and/or external shocks. Though external events are not the only explanations of the escape from the trap, they certainly played a role in giving the "big push" to investment in capital that allowed the escape from the malthusian stagnation towards the Modern Regime. This is clearly expressed for instance by Becker et al. (1990, S33): "*We believe that the West's primacy, which began in the XVII century was partly due to a `` lucky'' timing of technological and political changes in West*". It is therefore of interest to look at those, among the possible "escape mechanisms", which are internal to (4.1.2). Straighforward considerations show that i) domestic policies aimed at increasing the saving rate, ii) increases in the technology index Q, iii) increases in the technological parameter β[11], are all factors that may have concurred in favouring the escape. All these factors indeed cause upward shifts of the m_1 curve. Consider more in detail, for instance, the effects of a policy aimed at increasing the saving rate. Three situations arise: a) the upward shift of m_1 is so low that the new curve, termed m_1^{**}, still intersects the $m_2(k)$ curve only once. In this case no escape from the trap occurs (though the economy experiences an increase of per-capita income at equilibrium); b) the increase in s is large enough so that three equilibria appear. In this case the escape occurs if history and "luck" allows k to skip beyond k_{med}; 3) the upward shift is large enough so that m_1 skips the hump in the m_2 curve. In this case the poverty trap equilibrium k_{low} is lost and the economy approaches the virtuous steady state k_{high}. Notice that the escape from the trap could also be allowed by a sufficiently long temporary policy of raising the saving rate.

A further escape factor may be represented by aids to development, through a policy of donation of capital. This only affects, compared to i),ii),iii), the initial conditions of the system while leaving unaffected its equilibrium structure. Hence such a policy can be successful only provided that the "assisted" economy lives in a parameter constellation for which three equilibria exist. If the donation raises k above k_{med}, the economy will approach the modern steady state k_{high}, where (probably) the escape has been succesfull. Vice-versa, if the under-developed economy lives in the malthusian one-equilibrium regime, then the assistance policy will be hopeless, unless it does not also affects the parameters of the assisted economy in order to allow the appearance of the richer equilibrium.

All the aforementioned factors may have acted in some way in order to favour the escape. In particular the dynamical role of saving has been explored by Strulik (1999), who has given some theoretical support to the fact that the mortality decline observed during the DT could lead to an increase in the saving rate as well. This motivates to consider s as an increasing function of k in (4.1.2). This could have acted to sufficiently raise the m_1 curve perhaps allowing first, the appearance of multiple equilibria, and subsequently, the disappearance of the poverty-trap equilibrium. In other words, the transitional process seems to have been endowed by an endogenous escape mechanism (having the mortality decline as its ultimate responsible), which seems well suited in order to explain why a subset of countries of the world, the rich ones namely, seemed to have never been involved in the marsh of the poverty trap.

Realistic convergence patterns

Let us consider the rate of change of k in the region (k_{med}, k_{large}) where escape from the trap has occurred. As fig. 2 shows, the current rate if change of k, g_k increases from zero (at k_{med}) up to a maximum at k^*, and then monotonically declines to zero again, as the system is attracted in the k_{high} equilibrium. The implications are noteworthy. In the original Solow's model g_k monotonically declines to zero as k increases toward its positive equilibrium value. This has led to the well-known

[11] Notice that the effects of changes in s and/or Q are "neutral" whereas those of β are k-dependent.

concept of convergence (Barro and Sala y Martin 1995): economies with lower capital per person are predicted to grow faster in per-capita terms, a fact which has often been denied on the empirical ground (see quotation from Chu in section 1). In model (4.1.2) economies which in the end escaped the malthusian trap do not exhibit convergent paths as in the Solow's model. Consider two economies A,B which escaped the trap, i.e. that after some external shock entered the (k_{med}, k_{high}) region with respective endowments $k_{poor} < k_{rich}$. These is a whole region in the set (k_{med}, k_{high}) in which the two economies initially diverge, i.e. the richer economy grows faster than the poorer one (the amplitude of such region depends on the the actual position of $k_{poor} < k_{rich}$). In other words richer countries become, in a first phase (the temporal lenght of which may be quite large) even richer. Only at a later stage the two economies will enter a phase of convergent dynamics similar to that of the Solow's model. We argue that the present mechanism offers the simplest explanation for the currently observed paths of rich versus developing countries.

Fig. 1. The three equilibria in the Solow's model with transitional dynamics.

Fig. 2. Realistic convergence in the CRS Solow's model with demographic transition

Remark 4 (Harrodian exogenous technical progress and non-existence of non-trivial equilibria)
When harrodian technical progress at the constant rate α is considered, jointly with a CRS Cobb-Douglas technology, i.e. $Y = Qe^{\alpha} L(t)^{\beta} K(t)^{1-\beta}$, the basic model (4.2.1) may be expanded in the following system in the per-capita variables $x=K/L$, $y=Y/L$

$$\frac{\dot{x}}{x} = s\frac{y}{x} - \delta - n(y) \quad ; \quad \frac{\dot{y}}{y} = \alpha + (1-\beta)\frac{\dot{x}}{x} \qquad (4.2.1)$$

It is easy to check that the previous system does not have non trivial equilibria in the per-capita variables (only states of balanced growth). Indeed for $\frac{\dot{x}}{x} = 0$ then $\frac{\dot{y}}{y} = \alpha > 0$. This appears to be a highly serious limitation of the CRS model in the explanation very long term economic growth, that strongly motivates the analysis of the Decreasing Returns to Scale framework of next section.

5. The demographic transition and the Solow's model: a more general framework with decreasing returns to scale

The last remark of the previous section suggests that the CRS framework is irrealistic when one deals with historical time scales, and therefore also for the demographic transition. Indeed the existence of CRS with a steadily growing technology has not at all been the rule in historical time. This suggests that Decreasing Returns to Scale (DRS) could provide a more faithful representation. In addition the DT has not occurred in a constant economic environment. As suggested by a large body of historical evidence (Livi-Bacci 1997, 1998; see also the discussion in section 2.1) a non-trivial investigation of the implications of the demographic transition would potentially imply the need to simultaneously endogenise all the relevant parameters of Solow's model:

1. The rate of technological change (α). As largely documented (Livi-Bacci 1998, and refs. therein) the rate of growth of technology progress has not been constant and positive since historical times, but has rather experienced a historical evolution from an initial value of zero, followed by a very slow increase, before the take-off of the modern era. This assumption has

been used for instance in Prskawetz et al. (2000), who assumed a logistic increase in $\boldsymbol{a}$ as per-capita human capital increased over time.

2. The saving rate: theoretical and empirical evidence suggests that the saving rate increased along with per-capita income during the DT. This has been modelled by Strulik (1999) by using an s-shaped relation between the saving rate and pre-capita income.
3. Returns to scale: a more reasonable solution is to assume decreasing returns to scale during most of history, but with endogenous technical coefficients of production possibly leading to CRS in the long term, i.e. denoting with $\boldsymbol{g},\boldsymbol{b}$ the technical coefficients of labour and capital in Cobb-Douglas type production functions: $\lim_{t\to\infty}(\boldsymbol{b}(t)+\boldsymbol{g}(t))=1$ (possibily with an increasing ratio β/γ).

5.1 A general neoclassical framework for the DT under DRS

We now introduce a more general framework for the study of the dynamical implications of the DT under the previously sketched more realistic assumptions: DRS, endogenous technical change, and endogenous saving.[12]. A similar framework is used in Strulik (1999) for the study of the DT in developing countries. The following production function is assumed:

$$Y = A(t)L(t)^{\boldsymbol{b}}K(t)^{g} \qquad Q>0,\ \ \boldsymbol{a}>0\ ;\ \boldsymbol{b}+\boldsymbol{g}-1<0 \qquad (5.1.1)$$

where A(t) denotes the level of technology. Standard Solow-type assumptions lead (see the appendix) to the following formulation in the per-capita variables *x=K/P,y=Y/P* where *P* is the total population size (or the total supply of labour)

$$\frac{\dot{x}}{x}=s(y)\frac{y}{x}-\boldsymbol{d}-n(y) \qquad (5.1.2a)$$

$$\frac{\dot{y}}{y}=\boldsymbol{a}(y)+\boldsymbol{g}\frac{\dot{x}}{x}-(1-\boldsymbol{b}-\boldsymbol{g})n(y) \qquad (5.1.2b)$$

in which the population term *n(y)* is always a positive humped function of per-capita income *y*, whereas both the saving rate *s* and the rate of technical progress $\boldsymbol{a}$ are taken as increasing and saturating (logistic-alike) functions of *y*. It is convenient for analysis to consider the variables *U=y/x*, and *y*. We obtain:

$$\frac{\dot{U}}{U}=\boldsymbol{a}(y)+(1-\boldsymbol{g})\boldsymbol{d}+\boldsymbol{b}n(y)-(1-\boldsymbol{g})s(y)U \qquad (5.1.3a)$$

$$\frac{\dot{y}}{y}=\boldsymbol{a}+g\left(\frac{\dot{y}}{y}-\frac{\dot{U}}{U}\right)-(1-\boldsymbol{b}-\boldsymbol{g})n(y) \qquad (5.1.3b)$$

where (5.1.3b) yields, once expanded:

$$\frac{\dot{y}}{y}=\boldsymbol{a}(y)-\boldsymbol{g}\boldsymbol{d}+\boldsymbol{g}s(y)U-(1-\boldsymbol{b})n(y) \qquad (5.1.4)$$

In order to have a full understanding of the results of the present section, it is useful to recall the following result concerning the basic neo-classical DRS model (i.e. model (5.1.3) for exogenously determined *s,n,*$\boldsymbol{a}$):

Proposition (neo-classical model with DRS**)**. Under constant *s,n,*$\boldsymbol{a}$ model (5.1.3) has a unique globally asynptotically stable state of balanced evolution in per-capita variables. The long term rate of change of per-capita variables is:

[12] We leave to future work the investigations of the effcts of a full endogenisation of the technical coefficients in the production function.

$$q = \frac{\alpha - (1-\beta-\gamma)n}{(1-\gamma)}$$

In particular balanced growth occurs for $\alpha - (1-\beta-\gamma)n > 0$, whereas balanced decay occurs in the opposite case. In absence of technical progress (α=0) the long term outcome is always balanced decay at the rate $q = -\frac{(1-\beta-\gamma)n}{(1-\gamma)}$

5.2 Only population growth is endogenous

Rather than starting from the investigation of the general properties of (5.1.3), let us proceed in a hierarchical manner, by starting from the basic situation in which only population effects are considered, i.e. $n=n(y)$ according to the DT, whereas s,α=const. System (5.1.3) yields, for $U>0$

$$\dot{U} = U\left(\alpha + (1-\gamma)\delta + \beta n(y) - (1-\gamma)sU\right) \qquad (5.2.1a)$$

$$\dot{y} = y\left(\alpha - \gamma\delta + \gamma sU - (1-\beta)n(y)\right) \qquad (5.2.1b)$$

The equilibria

The equilibrium analysis is easily performed via the non trivial nullclines of the system, given by the lines

$$U = \frac{\alpha + (1-\gamma)\delta + \beta n(y)}{(1-\gamma)s} = h_1(y) \quad ; \quad U = \frac{\gamma\delta - \alpha + (1-\beta)n(y)}{\gamma s} = h_2(y) \qquad (5.2.2)$$

The curves $h_1(y)$, $h_2(y)$ inherit the humped shape of n(y). Let us consider the standard case of stationary population at both the beginning and the end of the transition: $n(0)=n(\infty)=0$.[13] In this case $h_1(0)>h_2(0)$, and $h_1(\infty)>h_2(\infty)$. There are two possibilities: a) the curve $h_1(y)$ always lies above $h_2(y)$ (no equilibria), or b) the curve $h_2(y)$ intesects twice $h_1(y)$. Case a) occurs for $\alpha>(1-\beta-\gamma)n_{Max}$, where n_{Max} is the maximal growth rate attained by the population during the DT, whereas b) occurs in the opposite case, $\alpha>(1-\beta-\gamma)n_{Max}$. Therefore no equilibria exist in case a), whereas exactly two equilibria exist in case b). In this latter case let us denote the equilibria by E_1,E_2, where E_1 is the "poor" equilibrium with smaller per-capita income. It is easy to show (see appendix 2) that E_1 is always LAS, whereas E_2 is always unstable. The nullclines and the directions of motion of system (5.2.1) are drawn in fig. 3 for case b). To fully understand the dynamics of the system it is also useful to look for states of balanced growth, i.e. asymptotic states characterised by exponential growth of per-capita variables at the same constant rate q, and therefore by an asymptotic constant ratio $U^*=H$. This leads to the following system in the quantities (H,q)

$$\alpha + (1-\gamma)\delta + \beta n(\infty) - (1-\gamma)sH = 0 \qquad (5.2.3a)$$

$$q = \alpha - \gamma\delta + \gamma sH - (1-\beta)n(\infty) \qquad (5.2.3b)$$

quickly giving:

$$q = \frac{\alpha - (1-\beta-\gamma)n(\infty)}{1-\gamma} \quad ; \quad H = \frac{\alpha + (1-\gamma)\delta + \beta n(\infty)}{(1-\gamma)s} \qquad (5.2.4a,b)$$

It is easy to show that such a state of balanced growth is locally asymptotically stable. The inspection of fig. 3 clarifies the dynamics in the more interesting case b). Notice that the asymptotically constant portion of the h_1 curve represents the radius of the balanced growth state.

[13] Some differences would arise if $n(0)>0$ and/or $n(\infty)>0$ (or even $n(\infty)<0$, according to the second demographic transition). As the purpose of this paper is not taxonomical we postpone the analyses of the minor subcases to future work.

The reader will easily identifiy the basins of attraction of the stable malthusian equilibrium E_1, and of the state of long term balanced growth.[14]

Fig. 3. Equilibria and direction of motion for the neo-classical DRS model with DT. Case b) (two equilibria);

Let us summarise the main results concerning the neo-classical model (5.2.1) with DT by the following:

Proposition. In the neo-classical model (5.2.1) with DT, given the technological ratio γ/β, two main outcomes are possible: a) the rate of change of technology (α) is so large[15] to always absorb the population pulge observed during the DT. In this case the system always attains a long term state of balanced growth, at the rate $q = \dfrac{\boldsymbol{a} - (1 - \boldsymbol{b} - \boldsymbol{g})n(\infty)}{(1-\boldsymbol{g})}$; b) the growth of technology is not sufficiently fast to absorb the population pulge. In this case two equilibria E_1,E_2 are possible, and just depending on initial conditions, the economy will be attracted in the malthusian poverty trap E_1, or will attain a a long term state of balanced growth.

Remark 5. The previous result suggests that, contrary to the CRS case, the DRS framework allows the simultaneous coexistence of stable “historical” steady states with “modern” long term stable states of balanced growth, characterised by different basins of attraction greater. More generally the DRS allows to explain in a unified model both the malthusian stagnation and the modern growth (impossible to capture by the CRS model).

5.3 Endogenous population and saving

The model now also considers both endogenous population and the saving rate. This latter is assumed to follow, according to the diffusionist argument, an s-shaped pattern of increase with per-capita income during the transitional era. This case has been considered by Strulik (1999) in his framework of demographic transition and Fixed Wage Regulation (FWR) in developing countries.[16] The non trivial nullclines are:

$$U = \frac{\boldsymbol{a} + (1-\boldsymbol{g})\boldsymbol{d} + \boldsymbol{b}n(y)}{(1-\boldsymbol{g})s(y)} = h_3(y) \quad ; \quad U = \frac{\boldsymbol{g}\boldsymbol{d} - \boldsymbol{a} + (1-\boldsymbol{b})n(y)}{\boldsymbol{g}s(y)} = h_4(y) \qquad (5.3.1)$$

In this more general case the nullclines are not necessarily humped, as in the previous case. Still reasoning under the standard “configuration” of the transition (*n(0)=n(¥)=0*), it holds *$h_3(0)>h_4(0)$*, and *$h_3(¥)>h_4(¥)$*, with *$h_3(0)>h_3(¥)$*, and *$h_4(0)>h_4(¥)$*. The theoretically interesting situations depend on the possibility that the initial population growth with per-capita income has been faster, or slower, compared to that in the saving rate. In the first case the curves $h_3(y)$,$h_4(y)$ are, in most situations, one-humped, as in section 5.2, whereas in the second case both curves will be, in most situations, monotonically decreasing in y. In terms of equilibria this implies that again, as in the previous section, two main cases are possible: a) no equilibria, when the rate of exogenous technical

[14] The region to the right of the unstable equilibrium E_2 and above the curve h_1 is positively invariant, as it is the region to the right of E_2 comprisen between the two curves.

[15] Alternatively one could reason in terms of the composite technological parameter $\alpha/(1-\beta-\gamma)$ versus the rate of growth of the population.

[16] It is to be pointed out that, however, the FWR has some serious drawbacks that makes it at most a special case of the model considered here.

progress is much larger compared to the the maximal rate of increase of the population during the transition, and b) two equilibria in the opposite case. The dynamical analysis shows results quite similar to those of section 5.2 (the poor malthusian state is usually LAS, whereas the second equilibrium E_2 is unstable) with a major new, which is the appearance of stable oscillations around the malthusian equilibrium. Such stable oscillations appear through a Hopf bifurcation of the malthusian equilibrium (see Appendix 2). The main results on the joint action of endogenous population and saving according to the mechanisms observed during the DT are reported in the following:

Proposition. In the neo-classical model (5.1.3) with DT and exogenously increasing saving rate the following main outcomes are possible: a) the rate of change of technology (α) is so large to always absorb the population pulge observed during the DT. In this case the system always attains a long term state of balanced growth, at the rate $q = \frac{\alpha - (1-\beta-\gamma)n(\infty)}{(1-\gamma)}$; b) the growth of technology is not sufficiently fast to absorb the population pulge. In this case depending on the initial conditions, the economy will remain in the malthusian poverty trap E_1, or will attain a a long term state of balanced growth. The permanence in the malthusian regime may occur through stationarity, or through stable oscillations. The appearance of stable oscillations is the consequence of a quick change in the patterns of saving at the beginning of the transition, which locally (but only locally) destabilises the malthusian equilibrium.

Remark 6. The latter part of the previous proposition is of interest for economic growth theory. It suggests, via the most standard growth model for long period change, namely the neo-classical DRS model, that the persistent oscillations that scholars in economic history have usually explained through purely exogenous arguments, plague or famine crises for instance, could have been just a part of the story. In fact the economic system was potentially able to lead to purely endogenous oscillations around the "poor" malthusian stagnation equilibrium, for instance via the mechanism embedded in the model of this section.

5.4 The joint role of productivity changes and population transition

The results of section 5.2 strongly motivate the investigation of the joint role of population and technological change, as motivated at the beginning of this section. In particular case a) in section 5.2 appears unrealistic, as history shows that hardly could technological change have been very fast at the beginning of the transition. This motivates the analysis of the neoclassical DRS model with $n=n(y)$ and $\alpha=\alpha(y)$, where $\alpha(y)$ is assumed to be, as the empirical evidence suggests (Livi-Bacci 1998) an increasing (initially very slowly) and saturating (as in Praskawetz et al. 2000) function of per-capita income. The analysis suggests that most results of section 5.2 are preserved if we still rely on the standard assumptions already used, and add the further assumption that, as suggested by most historical evidence, the increase in the pace of growth of productivity temporally preceeded population growth. In this case we again have the coexistence of a locally stable malthusian equilibrium, with an unstable intermediate equilibrium, with a locally stable regime of balanced growth. Compared to section 5.3 no persistent historical oscillations induced by productivity are possible (see the appendix). On the other side other equilibrium phenomena are possible: for instance if we assume that the population increase temporally preceeds, rather than following, the increase in technology, a third equilibrium may appear, which seems to be of certain interest for instance for the study of the DT in the developing world. Since we are not aiming at a taxonomical outcome, we postpone the analysis of these many interesting special cases to future work.

6. Balanced growth versus growth with cycle

A major remark on the developments of the previous sections concerns the fact that the present realistic variants of the descriptive neo-classical model, though generating several interesting effects, among which steady oscillations around the malthusian stagnation equilibrium, are not capable of reproducing a major stylised fact of modern economies, namely steadily oscillating patterns of balanced growth. In our recent work on the role of age structure and time delay in demo-economic models, we have showed that population age structure, via the the time-lag between the response of fertility to current living conditions, and the time when the resulting births enter the labour force, can be a major source of oscillations (and therefore also of patterns of steadily oscillating balanced growth in descriptive neoclassical models). In particular we have found that age structure may force persistent oscillations in both the Solow's 1956 and the Goodwin's 1967 model, and that, moreover, even simplified representation of age structure, based on time-delays, may generate growth with cycle in the neo-classical model (Fanti and Manfredi 1999, Manfredi and Fanti 2000). We feel that the "completion" of long term growth models, as those discussed in the present paper, by further realistic demographic features, such as the age structure of the population, is a highly noteworthy are of further inquiry.

7. Conclusive remarks

The present paper critically discusses several aspects related to the modelling of the demographic transitions, and tries to offer an overview and some new results and perspectives on the role played by descriptive Solow-type models as interpretative tools of the transition. Despite the general view that the model of Solow is very poor in explaining transitional phenomena, we have tried to show that, once adequately equpped with sound (and not only descriptive, but also optimal, as in section 3.1) hypotheses on transitional patterns, both the traditional Solow-type model with Continuous Returns to Scale (CRS), and especially the descriptive neo-classical model with Decreasing Returns to Scale (DRS), can still offer important insight on long term demo-economic growth. In particular the basic Solow's 1956 CRS model, once enriched with an assumption on the dynamics of the supply of labour mimicking the DT, offers interesting insights on the interrelationships between population dynamics and poverty traps, and a simple, but interesting generalisation of the notion of convergence. In addition to these nice properties of CRS models, our neo-classical DRS framework also exhibit (contrary to CRS) the coexistence of stable poor "malthusian stagnation" equilibria with stable states of "modern" long term balanced growth, as historically observed. This feature suggests that DRS models, rather than CRS, are optimal candidates as unified models of "very-long" term economic growth.

References

Azariadis C.-Drazen A., Threshold externalities and economic development, *Quarterly Journal of Economics*, 105, pp. 501-526.

Barro R.J., Sala-i-Martin X. (1995), Economic growth, Mc Graw Hill, New York.

Becker G., Family Economics and Macro Behaviour, *American Economic Review*, 7, 1988, pp. 1-78.

Becker G.S., Barro R.J. (1988), A reformulation of the economic theory of fertility, *Quarterly Journal of Economics*, 103,1,1-25.

Becker G.S., Murphy K.M., Tamura R. (1990), Human capital, fertility, and economic growth, *Journal of Political Economy*, 98,5,S12-S37.

Benhabib J., Perli R. (1994), Uniqueness and Indeterminacy: On the Dynamics of Endogenous Growth, *Journal of Economic Theory*, 63, 113-142.

Blanchet D. (1994), Modélisation demo-economique", Presse Univ. France, Paris.

Browning M. (1992), Children and household economic behaviour, *Journal of Economic Literature*, 30, 1434-1475.

Chesnais J. (1992), The demographic transition, Oxford University Press, Oxford.

Chu C.C.Y. (1998), Population dynamics. A new economic approach, Oxford Univ. Press, Oxford.

Cigno A., Growth with Exaustible Resources and Endogenous Population, *Review of economic Studies*, XLVIII, 1981, pp.281-287.
Cigno A., Consumption vs. Procreation in Economic Growth, in Steinmann G. (ed.), *Economic Consequences Change of Population in Industrialised Countries*, Springer, Berlin, Heidelberg, New York, 1984a, pp.29-49.
Cigno A., Further Implication of Learning-by-doing: The effect of population on Per-Capita Income, *Bulletin of Economic Research*, 36(2), 1984b, pp.97-108.
Dahan M., Tsiddon D. (1998), Demographic Transition, Income Distribution, and Economic Growth, *Journal of Economic Growth*, 3, pp. 29-52.
Dyson T., Murphy M., The Onset of Fertility Transition, *Population and Development Review*, 11, 1985, 399-440.
Erlich I., Lui F.T., (1991), Intergenerational Trade, Longevity and Economic Growth, *Journal of Political Economy*, 99, pp.1029-1059.
Fanti L., Manfredi P. (1999), Labour supply, time delays and demoeconomic oscillations in a Solow-type growth model, WP 136, Dipartimento di Statistica e Matematica Applicata all'Economia, Univ. Pisa, submitted.
Galor O., Weil D.N. (1999), From Malthusian stagnation to modern growth, Discussion Paper 2082, CEPR.
Galor O., Weil D.N. (1999), To Malthusian Stagnation to Modern Growth, *American Economic Review*, 89, 150-154.
Galor O., Weil D.N. (2000), Population, Technology and Growth: From the Malthusian Regime to the Demographic Transition and Beyond, *American Economic Review*, forthcoming.
Galor, O., Weil D.N. (1996), The Gender Gap, Fertility, and Growth, *American Economic Review*, 86 (3), 374-387.
Kremer M. (1993), Population Growth and Technological Change: One Million B.C. to 1990, *Quarterly Journal of Economics*, 108, 681-716.
Hansen G.D., Prescott E.C. (1999), *Malthus to Solow*, Federal Reserve Bank of Minneapolis, Research Department Staff Report 257.
Kögel T., Prskawetz A.(2000), *Agricultural productivity growth and escape from the Malthusian trap*, Max-Planck Institute for Demographic Research, WP 2000-02, 2000.
Knodel J.A. (1977), Age patterns of fertility and the fertility transition: evidence from Europe and Asia, *Population Studies* 31, 2, 219-250.
Jones (1999), Was an industrial revolution inevitable? Economic growth over the very long run, WP7375, NBER, Cambridge, MA.
Livi-Bacci M. (1997), A concise history of the world population, Blackwell, Oxford.
Livi-Bacci M. (1998), La popolazione nella storia d'Europa, Laterza, Bari.
Lee R.D., Malthus and Boserup: a dynamic synthesis, in Coleman D.-Schofield R.(eds.), *The state of population theory: forward from Malthus*, Basil Blackwell, Oxford, 1986, pp.96-130.
Livi-Bacci, Massimo. *A Concise History of World Population,* second edition, translated by Carl Ipsen. Oxford: Blackwell, 1997.
Lucas R.E., *The Industrial Revolution: Past and Future*, Unpublished Manuscript, University of Chicago, 1998.
Manfredi P., Fanti L. (2000), Population heterogeneities, nonlinear oscillations and chaos in some Goodwin-type demo-economic models, WP 178 Dipartimento di Statistica e Matematica Applicata all'Economia, Univ. Pisa, Paper presented at: Second workshop on Nonlinear demography, Max Planck Institute for Demographic Research, Rostock, Germany, May 31-June 2, 2000.
Mincer J., Market prices, opportunity costs, and income effect, in Christ C. et al. (eds.), *Measurements in Economics*, Stanford, Stanford University Press, 1963.
Momota A., Futagami K. (2000), Demographic transition pattern in a small country, *Economic Letters*, 67, 231-237.

Nelson R.R. (1956), A theory of the low level equilibrium trap in underdeveloped economies, *American Economic Review*, 46, 894-908.
Nerlove M., Raut L.K. (1998), Growth models with endogenous population: a general framework, in Rosenzweig M.R., Stark O. (eds.), Handbook of population and family economics, North-Holland.
Niehans J. (1963), Economic growth with two endogenous factors, *Quarterly J. of Economics*, 77, 349-371.
Palivos T. (1995) 'Endogenous fertility, multiple growth paths, and economic convergence.' *Journal of Economic Dynamics and Control,* 19, 1489–1510.
Prskawetz A., Steinmann G., Feichtinger G. (2000), Human capital, technological progress and the demographic transition, *Mathematical Population Studies*, 7, 4, 343-363.
Razin A., Ben-Zion U. (1975), An Intergenerational Model of Population Growth, *American Economic Review*, 65 (5), 923-933.
Retherford R.D., Palmore J.A. (1983), Diffusion processes affecting fertility regulation, in Bulatao R.A., Lee R.D. (eds.), Determinants of fertility in developing coutries, vol. 2, 295-339, Academic Press.
Rosero-Bixby L., Casterline J.H. (1993), Modelling diffusion effects in fertility transition, *Population Studies*, 47,147-167.
Solow R.M. (1956), A contribution to the theory of economic growth, *Quarterly Journal of Economics*, 70,1, 65-94.
Strulik H. (1997), Learning by doing, population pressure, and the theory of demographic transition, *Journal of Population Economics,* 10, 285-298.
Strulik H. (1999),Demographic transition, stagnation, and demoeconomic cycles in a model for the less developed economy, *Journal of Macroeconomics*, 21, 2, 397-413.
Tamura R. (1996), *Human Capital, Specialization, and Industrial Revolution*, Clemson University, mimeo.
Yip C.K., Zhang J. (1997), A simple endogenous growth model with endogenous fertility: Indeterminacy and uniqueness, *Journal of Population Economics*, 10, pp.97-110.

Appendix 1. The reference framework. A Solow-type model with Cobb-Douglas technology and exogenous technical progress

Consider the following production function with arbitrary returns to scale

$$Y = A(t)L(t)^{\beta} K(t)^{\gamma} \qquad Q > 0,\ \ \alpha > 0\ ;\ \beta + \gamma - 1 \gtrless 0 \qquad \text{(A1.1)}$$

where A(t) denotes the level of technology. Let P=the total population size, and consider overall per-capita variables x=K/P, y=Y/P.[17] By the equilibrium condition $\dot{K} = sY - dK$ one gets:

$$\frac{\dot{x}}{x} = \frac{\dot{K}}{K} - \frac{\dot{P}}{P} = s\frac{Y}{K} - \delta - n = s\frac{Y}{P}\frac{P}{K} - \delta - n = s\frac{y}{x} - \delta - n \qquad \text{(A1.2)}$$

i.e.

$$\frac{\dot{x}}{x} = s(y)\frac{y}{x} - d - n \qquad \text{(A1.3)}$$

where the possibility of endogenous saving has been considered. Moreover

[17] The distinction between variables y=Y/P and the traditional Y/L is formally unnecessary here but it is useful in more geenral contexts.

$$\frac{\dot{y}}{y}=\frac{\dot{Y}}{Y}-\frac{\dot{P}}{P}=\frac{\dot{Y}}{Y}-n \qquad \text{(A1.4)}$$

From the production function one obtains

$$\frac{\dot{Y}}{Y}=\frac{\dot{A}}{A}+\beta\frac{\dot{L}}{L}+\gamma\frac{\dot{K}}{K}=\alpha(t)+\beta\frac{\dot{L}}{L}+\gamma\frac{\dot{K}}{K} \qquad \text{(A1.5)}$$

where α(t) denotes the (possibly endogenous rate of techological change). Assuming $\frac{\dot{L}}{L}=\frac{\dot{P}}{P}=n(y)$ we have

$$\frac{\dot{Y}}{Y}=\alpha+\beta n(y)+\gamma\frac{\dot{K}}{K} \qquad \text{(A1.6)}$$

It follows

$$\frac{\dot{y}}{y}=\left(\alpha+\beta n(y)+\gamma\left(\frac{\dot{x}}{x}+n(y)\right)\right)-n(y)=\alpha+\gamma\frac{\dot{x}}{x}-(1-\beta-\gamma)n(y) \qquad \text{(A1.7)}$$

We have therefore derived a 2-dimensional system in the per-capita variables (x,y):

$$\frac{\dot{x}}{x}=s(y)\frac{y}{x}-\delta-n(y) \qquad \text{(A1.8a)}$$

$$\frac{\dot{y}}{y}=\alpha+\gamma\frac{\dot{x}}{x}-(1-\beta-\gamma)n \qquad \text{(A1.8b)}$$

This model is very similar to that derived by Strulik's (1999) (the only difference that the y equation is multiplied there by the constant $(1-\beta)^{-1}$).

Remark. The latter formulation holds for general patterns of returns to scale. In the special CRS case it boils down to

$$\frac{\dot{x}}{x}=s(y)\frac{y}{x}-\delta-n(y) \qquad \text{(A1.9a)}$$

$$\frac{\dot{y}}{y}=\alpha+\gamma\frac{\dot{x}}{x} \qquad \text{(A1.9b)}$$

The latter formulation has been used in section 4.2.

Appendix 2. Local stability analysis of the equilibria in the DRS neo-classical model with DT

Consider the form of the general model (5.1.3) relevant for local stability analyses

$$\frac{\dot{U}}{U}=\alpha(y)+(1-\gamma)\delta+\beta n(y)-(1-\gamma)s(y)U = f_1(U,y) \qquad \text{(A2.1a)}$$

$$\frac{\dot{y}}{y}=\alpha(y)-\gamma\delta+\gamma s(y)U-(1-\beta)n(y)= f_2(U,y) \qquad \text{(A2.1b)}$$

The Jacobian matrix is

$$J=\begin{pmatrix} f_1+U\frac{\partial f_1}{\partial U} & U\frac{\partial f_1}{\partial y} \\ y\frac{\partial f_2}{\partial U} & f_2+y\frac{\partial f_2}{\partial y}\end{pmatrix}=\begin{pmatrix} f_1-(1-\gamma)s(y)U & U\left(\alpha'(y)+\beta n'(y)-(1-\gamma)s'(y)U\right) \\ \gamma y s(y) & f_2+y\left(\alpha'(y)-(1-\beta)n'(y)+\gamma U s'(y)\right)\end{pmatrix}$$

In particular at all non trivial equilibrium points it holds

$$J=\begin{pmatrix} -(1-\boldsymbol{g})s(y)U & U\left(\boldsymbol{a}'(y)+\boldsymbol{b}n'(y)-(1-\boldsymbol{g})s'(y)U\right) \\ \boldsymbol{g}ys(y) & y\left(\boldsymbol{a}'(y)-(1-\boldsymbol{b})n'(y)+\boldsymbol{g}Us'(y)\right) \end{pmatrix} \qquad \text{(A2.2)}$$

Let us consider the stability properties of the "poor" malthusian equilibrium E_1 which, under standard assumptons, is a common feature of all the DRS models considered in section 5 of the paper.

A) In the pure population model (section 5.2) one simply gets

$$J_1=\begin{pmatrix} -(1-\boldsymbol{g})s_1U_1 & U_1\boldsymbol{b}n'(y_1) \\ \boldsymbol{g}ys_1 & -(1-\boldsymbol{b})n'(y_1)y_1 \end{pmatrix} \qquad \text{(A2.3)}$$

Since *n'(y)>0* in the malthusian state E_1, then $Tr(J_1)=-\left((1-\boldsymbol{g})sU+(1-\boldsymbol{b})n'(y)y\right)<0$, whereas $Det(J_1)=sUn'(y_1)y_1(1-\boldsymbol{b}-\boldsymbol{g})>0$, showing that E_1 is always LAS. Similarly, since *n'(y)<0* at E_2, this shows that E_2 is always unstable.

B) In the model with endogenous population and saving (section 5.3) one has

$$J=\begin{pmatrix} -(1-\boldsymbol{g})s(y)U & U\left(\boldsymbol{b}n'(y)-(1-\boldsymbol{g})s'(y)U\right) \\ \boldsymbol{g}ys(y) & y\left(-(1-\boldsymbol{b})n'(y)+\boldsymbol{g}Us'(y)\right) \end{pmatrix} \qquad \text{(A2.4)}$$

The last expression shows that a Hopf bifurcation occurs for some parameter constellations at the malthusian equilibrium. In fact the Trace may become negative for some paramer constellation, especially when the saving rate is quickly increasing at the beginning of the transition era, whereas $Det(J_1)=ys(y)U(1-\boldsymbol{b}-\boldsymbol{g})n'(y)>0$.

C) Exogenous population and technology. It holds

$$J=\begin{pmatrix} -(1-\boldsymbol{g})sU & U\left(\boldsymbol{a}'(y)+\boldsymbol{b}n'(y)\right) \\ \boldsymbol{g}ys & y\left(\boldsymbol{a}'(y)-(1-\boldsymbol{b})n'(y)\right) \end{pmatrix} \qquad \text{(A2.5)}$$

Therefore a Hopf bifurcation of the malthusian state is not possible in this case. In fact at the poor malthusian equilibrium E_1 it holds: $Tr(J)=-(1-\boldsymbol{g})sU-y\left((1-\boldsymbol{b})n'(y)-\boldsymbol{a}'(y)\right)$, and $Det(J)=sUy\left((1-\boldsymbol{b}-\boldsymbol{g})n'(y)-\boldsymbol{a}'(y)\right)$.

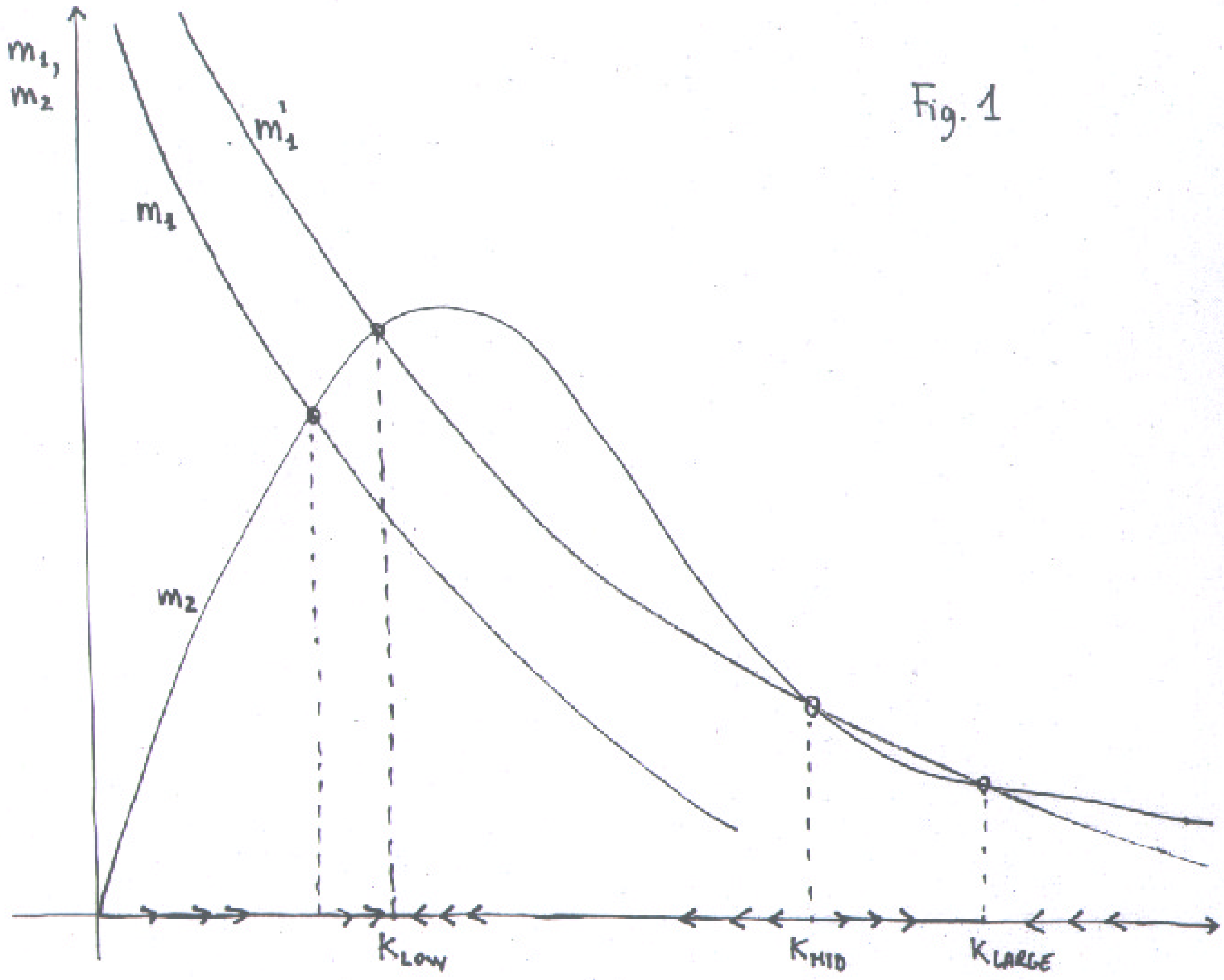
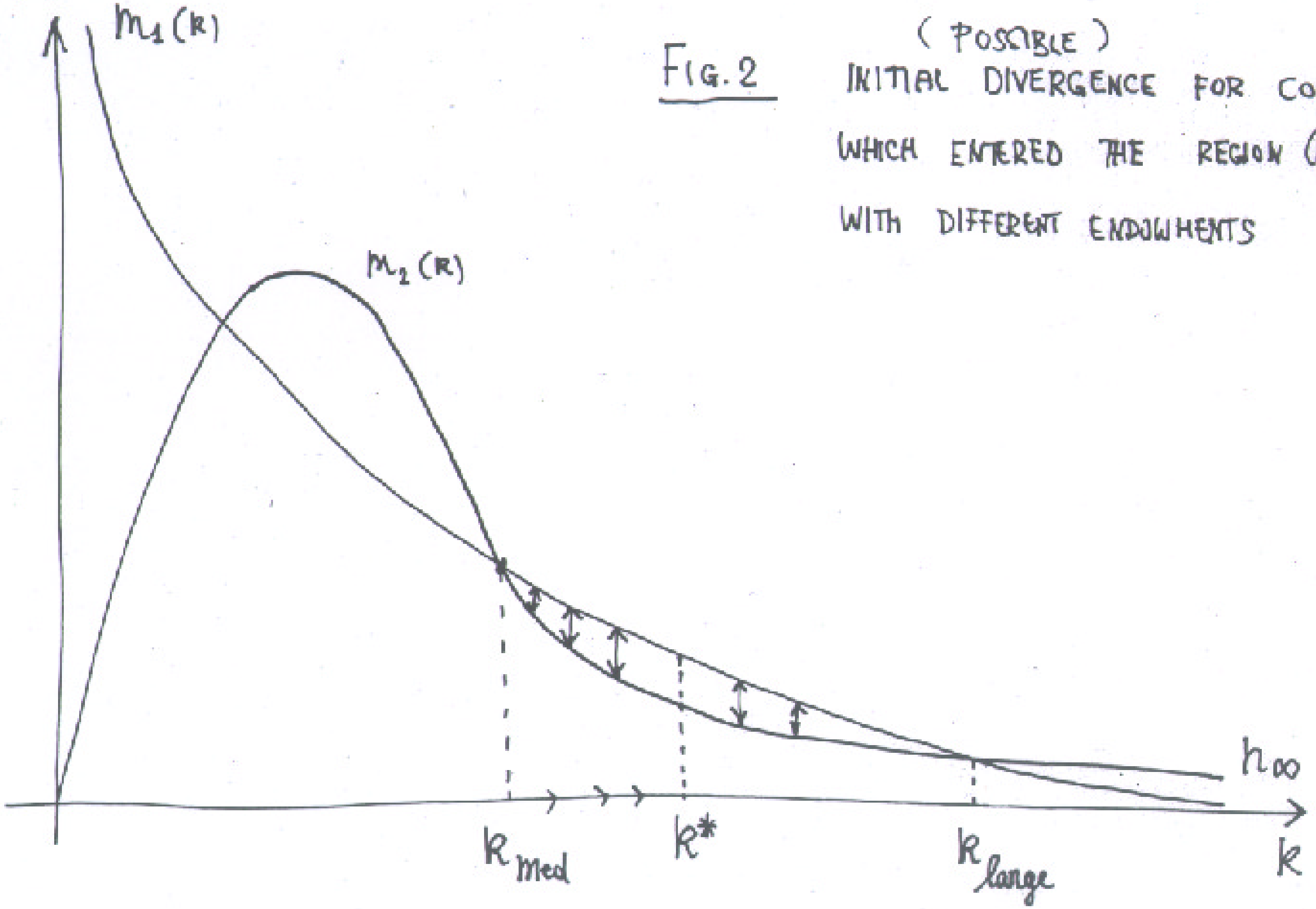
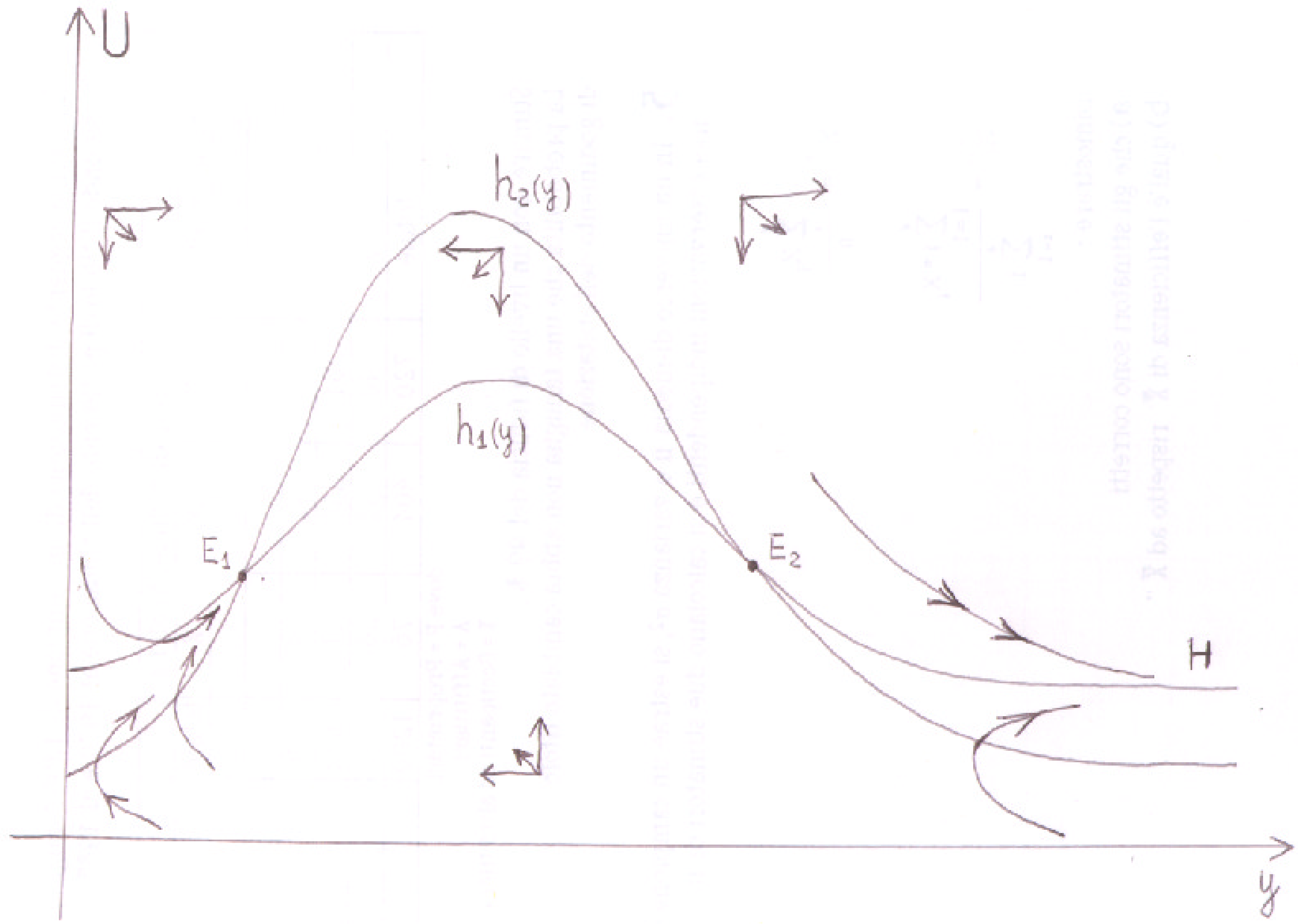


Fig. 1

FIG. 2 (POSSIBLE) INITIAL DIVERGENCE FOR COUNTRIES WHICH ENTERED THE REGION (k_{med}, k_{large}) WITH DIFFERENT ENDOWMENTS





U
$h_2(y)$
$h_1(y)$
E_1
E_2
H
y