Population, unemployment, and economic growth cycles: a further explanatory perspective.

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ABSTRACT

This paper investigates the dynamic interactions among economic growth, unemployment, income distribution and population growth. The model combines rational behaviour of agents, and pro…ts as the crucial determinant of the accumulation of …rms with endogenous fertility and unemployment. In particular the supply of labour is determined by micro-founded fertility choices of the individuals. We …rst demonstrate, consistently with the empirical evidence, the existence of a positive income growth trend with cyclical oscillations, therefore providing an alternative explanation of the relation between growth and cycle. Moreover the model seems to provide interesting insight on the relation between unemployment and growth. Up to now results available in the literature have always found a negative relation between unemployment and growth (although it should be mentioned the exception of the positive relation arising in Schumpeter's "creative" disruption). On the contrary, we …nd a twofold action of unemployment (via its e¤ects on population which is the engine of growth) on economic growth: this can be both positive or negative depending on the relative level of the cost of childrearing of employed and unemployed persons, and the level of unemployment bene…ts. This allows us argue that an increase of the unemployment bene…t - as it has occurred in recent years in many countries as for example France and Spain - could lead to wide demo-economic ‡uctuations and to a positive e¤ect of the unemployment on economic growth.

1

Introduction

A stylised fact of economic growth is the existence of a positive income growth trend with cyclical oscillations. Despite this evidence, however, both the "old" and the "new" theories of growth have generally neglected this important relation.[1] As it is known, the traditional neoclassical Solow's model (which, although the recent literature has been concentrated on the "new growth theory", still shows an excellent econometric validity- see Barro-Sala-y-Martin ,1995) depicts a world with a globally stable positive growth equilibrium, but also shows two restrictive features for what concerns the object of this paper: 1) it does not take into account the stylised fact of the existence of unemployment, which is generally not only positive but also strongly ‡uctuating; 2) in such a model ‡uctuations can be induced only by stochastic disturbances. Though the original model assumes inelastic labour supply, also its subsequent variants in which the labour supply is made variable have neglected unemployment as a critical dynamical variable.

As for the "new growth theory" some authors have recently pointed out the possibility of the occurrence of endogenous oscillations in simple variants of well-known models. For instance Benhabib-Perli (1994) have developed a four-dimensional (reduced to 3D after an appropriate change of variables) dynamical version of the two-dimensional famous Lucas' model (1988) by assuming elastic labour supply, but unfortunately the authors are more concentrated on the problem of 'indeterminacy' of the equilibrium rather than on the integration of cycle and growth as explicitly stated: " we have chosen to stress the indeterminacy results, since they are more relevant for the diverse growth experiences of some countries...., so we do not pursue the question of cycles any further.", p.124).

Another feature of the neoclassical growth theory has been the lack of consideration of the endogeneity of the fertility and more in general of the labour supply, until very recent papers: however also the models of the new growth theory in which population growth is endogenous (i.e. Becker-Murphy-Tamura (1990), Galor-Weil (1999), Jones (1999)) have assumed full employment and neglected the feature of the growth cycles.

We think that a very representative scenario of the economy must take account for the existence of unemployment and that according to the most part of the recent labour market theories (i.e. the 'unionised labour market' theory, the 'search'theory and the 'e¢ciency wage' theory) the wages are dynamically linked with the unemployment; moreover the fertility choices are dependent of the economic variables.

In order to surmount the above sketched limits we determine the supply of labour through the micro-founded fertility choices of the individuals and assume sluggishly adjusting, non market-clearing real wages. By doing so, we are able to obtain another further explanation of the integration between growth and cycle.

This paper may be seen as a contribution to the growth literature aiming to develop a Solovian growth model with sluggishly adjusting, non market-clearing real wages and endogenous fertility. Our economic assumptions are very general: 1) the economy is populated by maximising pro…t …rms with a technology characterised by an elasticity of subsitution less than one and abour augmenting productivity and by maximising individuals choosing their optimal fertility rate; 2) the wage determination does not depend directly on the production function but in a very general way (i.e according to a real Phillips-Lipsey curve) can depend on a bargaining context as well as "voluntary unemployment" in a walrasian context. Therefore we have combined the neoclassical growth theory à la Solow - where unemployment cannot exist - with those theories of the labour market which predict the onset of a positive 'naturale' rate of unemployment as entirely determined by the real conditions prevailing in the labour market (Friedman, 1975, p.161).

For what concerns the dynamical features of our economy we remark: 1) we have reproduced a typical feature of the real economies: a trend in the (endogenously determined) growth rate which also displays ‡uctuations, which, otherwise, are neglected in the old and new growth models; 2) the equilibrium point is not globally stable

[1] A famous exception is the Goodwin's model (1967), but the absence of both labour-capital substituibility and pro…t-maximising …rms in such a model have been considered as a too strong shortcoming.

(as in the neoclassical growth model): the level of the unemployment bene…ts is responsible for ‡uctuations and for strong destabilisation when it raises beyond a some level but below a critical level a stable growth (as in neoclassical growth model) is possible. The same result can be obtained as the cost of childrearing either for unemployed persons or for workers varies (for example due to the policy choices for what concerns the nursery-schools).

Moreover this paper presents a simple framework for analyzing the role of population as engine of growth, and then it adds to the many recent growth models in which population matters (Jones, 1999). For what concerns the long run results, as an increase in the population growth rate, other things equal, leads to an increase in the growth rate of total income, an expected implication of this model is that subsidies in order 1) to reduce the cost of childrearing (for all the types of individuals) or 2) to raise the preferences for children of the individuals or 3) to sustain the unemployed persons incomes (via unemployment bene…ts) may positively a¤ect the long-run growth rate. But the main result is that a link between income growth and unemployment is established, and in particular this link can be either positive or negative (a relatively new result in the economic debate about the relation growth-unemployment) depending on the di¤erent cost of childrearing (for di¤erent types of individuals) and the level of unemployment bene…ts.

From a methodological point of view we can distinguish the e¤ect of an economic parameter in our model according to the following taxonomy: 1) e¤ects on the short run or the long-run growth rate; 2) e¤ects on 'birth and death' of the equilibria; 3) e¤ects on the (local and possibly global) stability 4) e¤ects on the other economic state variables of the system and on their mutual relations in the short and in the long run: i.e. the growth-unemployment or the growth - distribution relationships. In this paper we have mainly investigated the e¤ects generated by changes in the unemployment bene…ts, which in the recent past have been increasingly relevant in the economy as well as in the economic debate, and we have shown a role of these latter so far neglected in the literature.

The rest of the paper is organised as follows. In section 2 we present the model, by articulating the various components (…rms, individuals, labour market rules). Section 3 analyses the existence of the equilibria and their stability properties, with the unemployment bene…ts being a particular case of parametric discussion. Section four investigates the long run growth properties of the model, concentrating about the emerging unemployment -growth relationship.In section …ve we explain how the simulations are carried out and include the numerical examples. Finally section six closes the paper with a review of the main conclusions.

2 The model of the economy

We have a closed, "real", economy populated by rational individuals and maximising …rms, and a labour market governed by a real wage bargaining system, represented by a very simple linear Phillips curve. The dynamics of this economy arises from the rate of accumulation, from the wage bargaining and from the population dynamics This economy displays the feature of long run growth in the total income. This feature, together with the endogeneity of population growth, allows to consider this model as an endogenous growth model. In addition this model shows cyclical ‡uctuations of the growth around a constant trend.

2.1 The …rm

In a competitive market the …rms (capitalists) are pro…t maximising and use a CES technology, which as known can represent any possible elasticity of substitution and embed many other production functions (Cobb-Douglas, Leontief and so on) as its special cases.

$$Y = c\left[zK^{-\theta} + (1-z)L_E^{-\theta}\right]^{-\frac{1}{\theta}}; \quad 0 < z < 1; \quad 0 < \theta < 1 \tag{1}$$

where z is a distribution parameter (recall that z is not a distributive share, unless in the special case in which $\theta \to 0$; corresponding to the Cobb-Douglas case) and $\eta = 1/(1+\theta)$ is the elasticity of substitution. In this paper we assume $\theta > 0$; equivalent to postulate a technology with a low degree of substitution between capital and labor[2]. The labour input is measured in efficiency units: $L_E(t) = L(t)\beta(t)$; where $L(t)$ is physical labour, whereas $\beta(t)$ represents the stock of knowledge or the labour augmenting technical progress. The function $\beta(t)$ can be exogenous or endogeneous (see Fanti-Manfredi, 2001).

The profit maximising firms hires labour until productivity of the marginal worker equals the real wage. By considering the production function in intensive forms, we obtain the following optimal factor demand ratio (see appendix) in terms of the distributive workers' share of the national income (V):

$$\frac{K}{L} = \left[\frac{(1-z)(1-V)}{zV}\right]^{-\frac{1}{\theta}}\beta = \phi(V)\beta \qquad \phi' > 0 \tag{2}$$

By assuming instantaneous equilibrium in the goods market, we have the equality saving-investment; total investment (I) is equal to profits (P) reinvested according to a fraction s_p by firms: $I = s_pP$. The rate of accumulation can be expressed in function of the distributive shares:

$$\frac{\dot{K}}{K} = \frac{I}{K} = \frac{IY}{YK} = \frac{s_pP}{Y}\frac{Y}{K} = [s_p(1-V)]\,c\left(\frac{1-V}{z}\right)^{\frac{1}{\theta}} \tag{3}$$

from which

$$\frac{\dot{K}}{K} = s_p c z^{-\frac{1}{\theta}}(1-V)^{\frac{1+\theta}{\theta}} \tag{4}$$

The accumulation rate shows that firms finance their investments by their profits income. This is also coherent with empirical evidence as well as the neo-keynesian theory of investment (Fazzari et al., 1988). This does not mean that the wage earners do not save at all, but only that they do not purchase shares of the firms. In fact they could save by purchasing public debt and if (as soon as) i) the public budget is balanced, ii) the public expenditure and taxation do not influence the firms' accumulation and iii) the firms do not purchase public debt, in this model - where the Say's law holds and the optimal fertility is independent of income - the saving (exogenously given as in Solow's model) of the individuals would be completely ininfluent. This situation, though simplicistic, is realistic enough in a country as Italy, where wage earners save, but the firms are owned by 'families' and are not 'public companies', and the savings of the individuals are largely held in public debt (the so-called "Bot people").

2.2 Employment status and the individuals fertility behaviour

In recent times there has been been a growing discussion on the possible roles of employment, and unemployment, as fertility determinants. Kalwij (2000) is a good references source. Kalwij himself suggests that for some developed countries, such as the Netherlands, the female employment status during life-cycle, is a major determinant of the presence and number of children in households: "employed women schedule their children later in life, and have fewer children compared to non-empoloyed women, holding educational attainment constant" (Kalwij 2000, 221). Despite this growing body of evidence there is still a lack of theoretical studies aiming at investigating the dynamical interaction between fertility and employment in economic macro-models. One possible reason for this state of affairs is the absence of theoretical frameworks, one exception is Goodwin's model (1967), having

[2] Rowthorn (1995,1999) reports a relevant quantity of empirical works supporting this assumption.

unemployment as a core variable [3]. In this section we develop a micro-founded framework of fertility decision taking the employment status into account.

In this framework there are two types of individuals: employed individuals and unemployed individuals. At every instant of time each individual (each family) determines the crude birth rate b[4], trading off between the consumption, c, and the cost of childrearing on the basis of well-behaved preferences (the preferences are the same for both types of individuals), given the income perceived (which is taken as given by individuals). The income of the employed individual is the wage earned per unit time (w), the income of the unemployed individual is the benefit of unemployment, which is assumed to be a constant fraction h (the so-called replacement ratio) of the wage. Also the cost of childrearing is different for the two types of individuals: it is reasonable to assume that rearing children is more expensive for an employed individual because of the opportunity-cost of the wage forgone during the time of care for children. We assume that the cost of childrearing is given by an exogenous constant fraction of the wage w for both types of individuals. Such a fraction includes both effective costs and opportunity costs. We denote such fractions as q for the workers and q_u for the unemployed individuals (so that qw and $q_u w$ define the real cost per child respectively for the workers and for the unemployed individual), and moreover that $q_u < q$, in that such costs embody the difference in the opportunity cost of the children.

The employed individual's optimization problem at each time t is given by

$$\max_{c_t, b_t} U(c_t, b_t) \tag{5}$$

The optimal choice of c and b is subject to the income constraint (as in Strulik, 1999)

$$c + qwb \leq w \tag{6}$$

Let us assume that the preferences are represented by the following standard log linear utility function:

$$U(c, b) = c^a (b)^{1-a}; \qquad 0 < a < 1 \tag{7}$$

Simple calculations provide the demand for children by employed individuals:

$$b_e = \frac{(1 - a)}{q} \tag{8}$$

which is to be interpreted as the fertility rate of the employed population. Similarly, by assuming that the preferences of unemployed individuals are represented by the same log linear utility function, the unemployed individual's optimization problem at each time t is given again by $\max U(c, b) = c^a (b)^{1-a}$; $a < 1$, subject to the income constraint $c + q_u wb \leq hw$. Simple calculations provide the demand for children by unemployed individuals:

$$b_u = \frac{(1 - a)h}{q_u} \tag{9}$$

denoting the fertility rate of the unemployed population. Since at every instant of time in the economy a fraction E of the individuals is employed and another fraction $(1 - E)$ is unemployed, the overall rate of fertility is:

[3] Manfredi and Fanti (2000) consider, within a Goodwin-type macro-economic model, the role of heterogeneity in fertility arising as a consequence of different employment status of individuals.

[4] In other words: the family determines the flow of birth per unit time, rather than the overall stock of children during lifetime, as done in overlapping generation models. This approach is nowadays well acknowledged in economics (Jones 2000, Momota and Fugatami 2000).

$$b = (1-a)\left(\frac{h(1-E)}{q_u} + \frac{E}{q}\right) = (1-a)\frac{qh(1-E)+q_uE}{qq_u} \tag{10}$$

In this case the fertility function is independent of the wage and dependent of the unemployment rate. Since in this macroeconomic model the production is always determined by the supply side (the Say's Law holds), the demand for consumption is ininfluent. The demand for children could seem as temporarily optimal, but not intertemporally optimal; however this kind of optimality could be sensible in that it appears reasonable to assume that "it is impossible for individuals to guess or it is too costly for them calculate what future generations may want to do...."(Day et al., 1989, 143). But there is a more rigorous reason to hold this static optimization: as claimed by Jones (1999) the present static optimization problem can be derived from a more general dynamic optimization problem by simply assuming that: 1) the utility depends on the flow of births rather than on the stock of children; 2) each individual faces a probability of death which depends on aggregate per capita consumption, which individuals take as given; "with these assumptions, the more standard dynamic optimization problem reduces to the sequence of static problems given above"(Jones,1999,p.5).

2.2.1 Some remarks on the fertility function

Since now on we definitively adopt the hypothesis that $q_u < q$. The fertility choice determines the birth rate as a function of several economic parameters: $b = b(E, q, q_u, h, a)$. In particular the fertility of the employed (b_e) exceeds or not that of the unemployed population (b_u), for $h < q_u/q = h_C < 1$ or $hq/q_u < 1$. This suggests that h could be a potentially critical parameter. In particular for any E, as h increases from 0 to 1 the overall fertility rate $b(E)$ is a monotonically increasing function of h (therefore implying that also the rate of change of the labour supply $n(E)$ is monotonically increasing), as clear from:

$$\frac{\partial b}{\partial h} = \frac{(1-a)}{qq_u}q(1-E) > 0 \quad \forall (h, E) \tag{11}$$

In other words: for fixed U an increase in h raises the contribution to fertility of unemployed while leaving unaffected that of employed. On the other side

$$\frac{\partial b}{\partial E} = \frac{(1-a)(q_u - qh)}{qq_u} \tag{12}$$

This shows that the overall fertility depends positively (negatively) on the employment rate when the ratio between cost of childrearing and benefit for the unemployed individual is greater (less) than the cost of childrearing of the workers. In other terms b is an increasing function of E only for $h < q_u/q$ (i.e. $b_e > b_u$). At $h = q_u/q$ a switch occurs and in the region $h > q_u/q$ the unemployment benefit is so large that b becomes a decreasing function of E (or an increasing function of the level of unemployment). Which is the interpretation of this reversion in the relation between b and E ? The answer is the following: for very large h the relative fertility of unemployed may become very high compared to that of employed (because they have a lower cost of childrearing), and therefore an increase in E, which reduces the fraction of the population which has higher fertility leads coeteris paribus to a decrease in fertility.

As a final remark, we notice that the endogenization of population allows for to remedy the limit implied in most traditional growth models which usually only consider the case of $n > 0$ for all parameter constellations. Here the population is free to grow or decay, as in the real world.

2.3 The labour market

The labour market is in disequilibrium and the wage dynamics is determined by a wage bargaining represented, for sake of simplicity, by a linear Phillips equation, saying that when unemployment is low then workers become more powerful and are able to claim higher real wages (and viceversa when unemployment is high):

$$\dot{w} = w(-\gamma + \rho E) \qquad 0 < \gamma < \rho \tag{13}$$

where w = the real wage, $E = L/N$ is the rate of employment, N = the total supply of labour, and $\gamma; \rho$ denote characteristic labour market parameters.

Obviously the above relation can also refer to a neoclassical labour market working according to a "marshallian" adjustment process, as in the interpretation dating back to Lipsey (1960), Phelps (1970) and Holt (1970), so that the eventual rate of unemployment equilibrium in our economy can represent a "natural rate of unemployment" and the unemployment can be considered "voluntary". The wage bargaining (or, in the other interpretation, the existence of "voluntary" unemployment) also leads to modifications of the distributive share.

A simple sight of the optimal factor demand ratio says to us which is the effect of the workers'share: when workers are able to obtain a larger distributive share, firms find it less profitable to hire workers and therefore switch away from labour to machinery.

2.4 The final model

The employment dynamics results from the following dynamical identity:

$$\frac{\dot L}{L} = \frac{\dot K}{K} - \frac{\dot \phi}{\phi} - \frac{\dot \beta}{\beta} \tag{14}$$

Since

$$\frac{\dot \phi}{\phi} = -\frac{1}{\theta}\left[\frac{(1-V)^{\cdot}}{(1-V)} - \frac{\dot V}{V}\right] = \frac{1}{\theta(1-V)}\frac{\dot V}{V} \tag{15}$$

one gets

$$\frac{\dot L}{L} = s_p(1-V)^{\frac{1+\theta}{\theta}} cz^{\frac{-1}{\theta}} - \frac{1}{\theta(1-V)}\frac{\dot V}{V} \tag{16}$$

The dynamical equation of the employment ratio follows from $\dot E/E = \dot L/L - \dot N/N = \dot L/L - n$ where $n = \dot N/N$ is the rate of growth of the labour supply. In this paper we disregard participation effects, and assume that n is simply defined as the difference between the fertility rate b and the mortality rate μ. Therefore:

$$n = b - \mu = \frac{(1-a)}{qq_u}[qh(1-E) + q_u E] - \mu = n(E) \tag{17}$$

yielding

$$\frac{\dot E}{E} = s_p(1-V)^{\frac{1+\theta}{\theta}} cz^{\frac{-1}{\theta}} - \frac{1}{\theta(1-V)}\frac{\dot V}{V} - n \tag{18}$$

Finally by the identity, $V = w/A$ where $A = Y/L_E$ it is possible to derive the dynamics of the wage share:

$$\frac{\dot V}{V} = \frac{\dot w}{w} - \frac{\dot A}{A} - \frac{\dot \beta}{\beta} = \frac{\dot w}{w} - \frac{1}{\theta}\frac{\dot V}{V} - \frac{\dot \beta}{\beta} \Rightarrow \frac{\dot V}{V} = \frac{\theta}{1+\theta}\left[\frac{\dot w}{w} - \frac{\dot \beta}{\beta}\right] \tag{19}$$

This dynamical economy is therefore described by the following two-dimensional model in terms of the employment rate, E, and the share of labour V:

$$\begin{aligned} \frac{\dot V}{V} &= \frac{\theta}{1+\theta}[-\gamma + \rho E - a_0] \\ \frac{\dot E}{E} &= s_p(1-V)^{\frac{1+\theta}{\theta}} cz^{\frac{-1}{\theta}} - a_0 - \frac{1}{\theta(1-V)}\frac{\dot V}{V} - \frac{(1-a)}{qq_u}[qh(1-E) + q_u E] \end{aligned} \tag{20}$$

3 Static and dynamical properties of the model

Define

$$
\begin{array}{l}
s_p c z^{-\frac{1}{\theta}}(1-V)^{\frac{1+\theta}{\theta}} = A g_1(V) \qquad A = s_p c z^{-\frac{1}{\theta}} > 0 \\
f(E) = -(a_0+\gamma) + \rho E \\
C_0 = \frac{\theta}{1+\theta} > 0
\end{array}
\tag{21}
$$

where the function $g_1(V)$, which satisfies $g_1(0) = 1, g_1(1) = 0$, is non-negative and strictly decreasing on $(0,1)$. We obtain the form

$$
\begin{array}{l}
\frac{\dot V}{V} = C_0 f(E) \\
\frac{\dot E}{E} = A g_1(V) - \frac{1}{\theta(1-V)}\frac{\dot V}{V} - (n(E)+a_0)
\end{array}
\tag{22}
$$

where $n(E)$ was defined in 17.Notice that the second member of (22) is not defined for $V = 0, V = 1$. For $V \neq 0$ one has

$$
\frac{\dot E}{E} = A g_1(V) - \frac{C_0}{\theta(1-V)} f(E) - (n(E)+a_0) \tag{23}
$$

By definition of E, V the state space is the set $[0,1] \times (0,1)$. Provided $0 < V < 1$ the previous initial value problem always admits a unique solution which is also meaningful, i.e. solutions starting from positive initial conditions stay positive forever.

3.1 The equilibria

In what follows we discuss the process of birth and death of equilibrium points and their stability as a one-parameter discussion, by using as the critical parameter the replacement ratio h. Equilibria of (23) are solutions of the system $\dot V = 0$; $\dot E = 0$. It is possible to show that the system admits up to three equilibria. Notice first that the system always admits the "zero" equilibrium point $P_0 = (0,0)$.[5] Moreover at most one strictly positive equilibrium $P_1 = (V_1, E_1)$ exists, as solution in $(0,1) \times (0,1)$ of the equations

$$
\begin{array}{c}
A g_1(V) - (n(E)+a_0) = 0 \\
C_0 f(E) = 0
\end{array}
\tag{24}
$$

One quickly gets $E_1 = (\alpha+\gamma)/\rho$ which is meaningful for $\alpha + \gamma < \rho$. Equilibrium values for V are solutions of

$$
g_1(V) = \frac{n(E_1)+a_0}{A} \quad \rightarrow \quad (1-V)^{\frac{1+\theta}{\theta}} = \frac{n(E_1)+a_0}{A} \tag{25}
$$

Therefore, meaningful equilibrium values for V require $0 < n(E_1) + a_0 < A$, i.e.: $-a_0 < n(E_1) < A - a_0$. By assuming that the previous condition is fulfilled one finds

$$
V_1 = 1 - \left(\frac{n(E_1)+a_0}{A}\right)^{\frac{\theta}{1+\theta}} = 1 - \left[\frac{n(E_1)+a_0}{cs_p}\right]^{\frac{\theta}{1+\theta}} z^{\frac{1}{1+\theta}} \tag{26}
$$

Clearly $\frac{\partial E_1}{\partial n} = 0$; and $\frac{\partial V_1}{\partial n} < 0$. We notice that very low values of h could lead to a negative rate of growth of the population and therefore prevent the existence of a meaningful equilibrium value for V in the event $n(E_1) + a_0 < 0$. Therefore for some parameter constellation meaningful equilibria occur only for $h > h_* > 0$, i.e. for h greater than a lower threshold h^* (details in the appendix). In particular notice that $V_1(h = h_*) = 1$. On the other side

[5] Correctly speaking, since $V > 0$, the point $E_0 = (0,0)$ "behaves" as an equilibrium: it may be considered an "extended" equilibrium point for reasons of continuity. In fact the limit

$$
\lim_{V \to 0} \frac{\dot V}{V} = c \lim_{V \to 0} f(U)
$$

exists and is finite.

very high values of h; pushing n upward, could prevent again a meaningful equilibrium in the wage share. This occurs for $n(E_1) + a_0 > A$. The latter inequality leads to a condition of the form h > h¤¤;where h¤¤ > h¤. If h¤¤ is between 0 and 1 then no meaningful equilibrium value of V exists for every h > h¤¤. Vice-versa, if h¤¤ is greater than one, than no h value exists giving $n(E_1) + a_0 > A$, and therefore a meaningful equilibrium value of V always exists. In particular notice that V_1 (h = h¤¤) = 0.

In conclusion: by using h as the critical parameter, one sees that a meaningful positive equilibrium E_1 generally exists only in a suitable window of values of the h parameter, namely h¤ < h < h¤¤. [6] In practical terms: a very large unemployment bene…t may cause the disappearance of the positive equilibrium, a fact that is con…rmed by the numerical simulation of next section. Finally, for large h values, an axis equilibrium $P_2 = (V_2; E_2)$ may exist, and have some interesting economic properties, in particular from a welfare perspective (Fanti and Manfredi 2000a,b; 2001). We postpone to future work the discussion of such properties because this paper concentrates on the "growth and cycle" topic, and cycles are obviuosly impossible around equilibria di¤erent from the positive one.

4 Features of the long run growth

A number of recent papers have sought to characterize the relation between growth and unemployment, but so far with many restrictive assumptions, as in the following summarised. We can distinguish four areas in the literature on the relation between growth rate and unemployment (see for a recent survey Pugno 1998): 1) there is a full 'dichotomy' between the two variables, either obviously in absence of unemployment (Solow, 1956 and all the subsequent variants) or in presence of constant positive unemployment (Layard-Jackman-Nickell, 1991), 2) the 'dichotomy' can be broken simply by generalising the production function according to a CES and allowing for an elasticity of subsitution less than one, but this result has been shown in a very partial analysis, assuming accumulation and growth exogenously given (Rowthorn, 1999);by passing we note that this result has been neglected in the subsequent literature and our paper, inter alia, con…rms it in a more general economic context; 3) the 'dichotomy' is overcome by showing that in the long-run equilibrium di¤erent rates of growth can corrispond to di¤erent rates of unemployment (Pissarides, 1990); but this analysis compares only di¤erent static situations, by assuming an exogenous constant growth; 4) the dichotomy is also broken in some endogenous growth model of the AK type - where obviously the growth is always driven by saving or investments - by allowing for some labour market elements (as the employment) to a¤ect the accumulation rate, but such models are restricted to the case of a speci…c technology (Cobb-Douglas) and to the presence in the labour market either of a speci…c unions' behaviour (Daveri-Tabellini, 1997) or speci…c search costs (Bean-Pissarides, 1993).

We have developed a simpli…ed but su¢ciently general model to be capable to encompass more speci…c theories as special cases, showing constant (or ‡uctuating around a costant trend) total income growth which is fully endogenous and moreover allowing for an endogenous determination of accumulation, population growth, employment and distribution (which latter for instance is exogenously given in all the models using Cobb-Douglas technology). This model also permits to derive the very interesting conclusion that the unemployment can favour as well as unfavour economic growth, so enriching the answer of the literature on this topic. In the steady-state this economy is endogenously growing at the following rate of growth of total output:

$$\frac{\dot{Y}}{Y}(P_1) = n \tag{27}$$

The factors a¤ecting the growth are neatly depicted in the following expression:

[6] Of course under some parameter constellations it could occur h¤ < 0 or h¤¤ > 1. In these cases the region of existence of E_1 would simply be of the type (0; h¤¤) or (h¤; 1).

$$g^* = \frac{\dot{Y}}{Y}\bigg|_{E^*,V^*} = a_0 + b\left(\underset{(-)}{a}\quad \underset{(-)}{q}\quad \underset{(-/+)}{q_u}\quad \underset{(+)}{h}\quad \underset{(-/+)}{E}\right)$$

Therefore the main theoretical result is that the relation between growth and unemployment rate has an ambiguous sign. We remark an interesting consequence: if in countries with high unemployment the governments raise sufficiently the unemployment benefits - in order to reduce the social costs of the high unemployment - then the high unemployment can imply a higher rate of total income growth (via population growth).

4.1 Stability and bifurcation of the positive equilibrium P_1

Consider the Jacobian at P_1

$$J(P_1) = J_1 = \begin{pmatrix} 0 & C_0\rho V_1 \\ A g_1'(V_1) E_1 & -\left(\frac{1}{\theta}\frac{1}{(1-V_1)}C_0\rho + n'(E_1)\right)E_1 \end{pmatrix} \tag{28}$$

By recalling that g is strictly decreasing over $(0,1)$ we have (we suppress the subfix 1 in the variables E, V for simplicity)

$$Tr(J_1) = -\left(\frac{1}{\theta}\frac{C_0\rho}{(1-V)} + n'(E)\right)E \quad ; \quad Det(J_1) = (-1)AC_0\rho g_1'(V)EV > 0 \tag{29}$$

As $Det(J_1) > 0$ for every parameter constellation, possible switches from stability to instability are totally governed by $Tr(J_1)$. Stability of P_1, requiring $Tr(J_1) < 0$, occurs for

$$\frac{1}{\theta}\frac{C_0\rho}{(1-V_1)} - \frac{1-a}{qq_u}(qh - q_u) > 0 \tag{30}$$

Therefore:

i) if $qh - q_u < 0$, i.e $h < q_u/q = h_C < 1$, i.e. simply when $b_e > b_u$ the E_1 equilibrium is always locally asymptotically stable (LAS). In words, instability never occurs when the fertility rate of employed individuals exceeds that of unemployed.

ii) if $qh - q_u > 0$, i.e $h > h_C$ ($b_e < b_u$),the E_1 equilibrium is not necessarily LAS, and instability may arise.

Notice that if $h_{**} < h_C$ then all h values for which a meaningful positive equilibrium P_1 exists fulfill the condition $h < h_C$ and stability always prevails. If $h_C < h_*$ then all h values for which a meaningful positive equilibrium E_1 exists fulfill the condition $h > h_C$, and instabilities may occur in the whole window. In the intermediate (and most interesting) case $h_* < h_C < h_{**}$ stability always prevails in window $h_* < h < h_C$ whereas instability may occur for $h_C < h < h_{**}$. Let us now characterise the conditions under which instability may arise. Instability prevails for

$$(qh - q_u)(1 - V_1) > \frac{qq_u}{1-a}\frac{C_0\rho}{\theta} \tag{31}$$

By using h as the "pivotal" parameter, the latter instability condition may be written as

$$Q(h) > F$$

where (stressing dependency on h)

$$\begin{gathered} Q(h) = (qh - q_u)(1 - V_1(h)) = q(h - h_C)(1 - V_1(h)) \\ 1 - V_1(h) = \left(\frac{n(E_1;h) + a_0}{A}\right)^{\frac{\theta}{1+\theta}} \quad ; \quad n(E_1;h) = \frac{(1-a)}{qq_u}[qh(1-E) + q_uE] - 1 \\ F = \frac{qq_u}{1-a}\frac{C_0\rho}{\theta} > 0 \end{gathered} \tag{32}$$

The previous instability condition has to be considered on the set $D : h_* < h < h_{**}$ where a positive equilibrium exists, and is meaningful ($0 < 1 - V_1(h) < 1$). As stability always prevails for $h < h_C$; one can only consider $Q(h)$ on the set $h_C < h < h_{**}$. Consider first the case $h_* < h_C < h_{**}$. As the function $(1 - V_1(h))$ is positive on D the function $Q(h)$ is positive for $h_C < h < h_{**}$. Moreover, the function $(1 - V_1(h))$ is strictly increasing in h, implying that $Q(h)$ is strictly increasing for $h > h_C$ being the product of two strictly increasing functions. Moreover (remember $V_1(h = h_*) = 1$, $V_1(h = h_{**}) = 0$), it holds

$$Q(h_C) = 0 \quad ; \quad Q(h_{**}) = (qh_{**} - q_u)(1 - V_1(h_{**})) = q(h_{**} - h_C) > 0 \tag{33}$$

where $Q(h_{**}) = q(h_{**} - h_C)$ denotes the sup of the function $Q(h)$ on the set $h_C < h < h_{**}$. Therefore, if

$$Q(h_{**}) = q(h_{**} - h_C) > \Phi \tag{34}$$

then exactly one bifurcation point h_H exists (and it satisfies $h_H > h_C$). Completely equivalent things occur in the case $h_C < h_*$.[7] We may summarise our results on the local stability properties of P_1 by the following proposition:

Proposition.

i) if $h < h_C$, i.e. if the replacement ratio does not exceed the critical value h_C.the positive equilibrium P_1 is LAS in all its existence domain D:

ii) if $h > h_C$ instability may arise. In particular, if the condition $q(h_{**} - h_C) < \Phi$ continues to prevail for every h, whereas for $q(h_{**} - h_C) > \Phi$ the system becomes unstable when h exceeds the threshold value h_H.

iii) the point $h = h_H$ is a Hopf bifurcation point.

Points i) and ii) of the propositions are demonstrated in the previous discussion. The proof of iii), namely the appearance of a Hopf bifurcation (for instance Guckenheimer and Holmes 1983) at $h = h_H$ is straightforward: at $h = h_H$ purely immaginary eigenvalues occur since at h_H it holds $Tr(J) = 0$ under $Det(J) > 0$: Moreover the test for nonzero speed gives

$$\left(\frac{d}{dh}(Tr(J))\right)_{h=h_H} = (-E)\frac{d}{dh}\left(\frac{C_0\rho}{\theta(1-V)^2}\frac{dV}{dh} - \frac{1-a}{q_u}\right)_{h=h_H} > 0$$

as dV/dh is a strictly decreasing function of h.

Remark on the instability condition. Let us consider the IFF condition for instability $q(h_{**} - h_C) > \Phi$ i.e. $q(h_{**} - h_C) > \frac{qq_u}{1-a}\frac{C_0\rho}{\theta}$, which may be put (see the appendix) in the form

$$A + \mu > \frac{(1-a)}{q} + K(\rho - a_0 - \gamma) + a_0 \tag{35}$$

where $K = 1/(1+\theta)$. The latter inequality allows to study the influence played by the various economic parameters intervening in the system in influencing the stability of the positive equilibrium P_1, by looking at whether increasing a given parameter leads to an increase in the first member and/or a decrease in the second one of the latter relation. One easily sees the following:

i) the fertility rate of the employed population by $(1 - a)/q$ always plays a stabilising role

ii) since $\rho(1 - E_1) = \rho - a_0 - \gamma$ one has $K\rho(1 - E_1) + a_0 = (1 + \theta)^{-1}(\theta a_0 + \rho - \gamma)$ showing that also increasing rates of exogenous Harrodian technical progress are stabilising

[7] One has to study stability of E_1 on D, where $Q(z)$ is again a non-negative and strictly increasing function of z. Since $V_1(z = z_*) = 1$ then

$$Q(z_*) = (qz_* - q_u)(1 - V_1(z_*)) = 0 \quad ; \quad Q(z_{**}) = q(z_{**} - z_C) > 0$$

leading to the same conclusions of the previous case.

iii) as far as the labour market parameters are concerned, ° plays a destabilising role (con…rming other works by the authors), whereas the action of ½ is stabilising

iv) as $A = A(\mu) = s_p c z^{\frac{¡1}{\mu}}$ is, other things being equal, a monotonically increasing function of μ, whereas $K = 1=(1+\mu)$ is a monotically decreasing function of μ; this suggests that large values of μ are necessary in order to destabilise. This well agrees with the fact that the Cobb-Douglas case (μ ! 0) is always a stable one (as can be easily cheched by the jacobian). The previous results are summarised in the following table:

$b_e = \frac{1 ¡ a}{q}$	a_0	½	°	μ
+	+	+	¡	¡

Table 1. Stabilising (+) or destabilising (-) role played by some of the

main economic parameters considered

5 Simulative evidence and working of the system

The fact to know that a Hopf bifurcation exists nothing says about the stability properties of the involved periodic orbit(s), i.e. it does not say whether the bifurcation is supercritical or subcritical (i.e. whether the periodic orbit is locally stable or unstable). Since the predictions of the Hopf theorem are "local" (they nothing say about global behaviours) we decided to investigate numerically the stability properties of the periodic orbits emerged via Hopf bifurcation of P_1 at the point $h = h_H$, and more generally to investigate the global properties of our model. The simulative evidence shows two remarkable facts: i) $h = h_H$ generates supercritical bifurcations (i.e. locally stable oscillations); ii) when h is relatively high the positive equilibrium disappears, con…rming our theoretical predictions. We illustrate the actual working of the model by a concrete example in which, just to reduce complexity, we sterilize the e¤ects of all the other parameters and concentrate only on the dynamical e¤ects of the replacement ratio parameter h. In the following experiments we set $a_0 = 0; \mu = 10; c = 0{:}1; h = 0{:}5;$ ° $= 0{:}1;$ ½ $= 0{:}12; a = a = 0{:}987; q = 0{:}2; q_u = 0{:}05{:}$ The simulation shows that by increasing the replacement ratio h the phase portrait of the system undergoes the following transformations: convergence to a globally stable node or focus ! convergence to a stable limit cycle ! possible 'explosion' of the orbits ! the equilibrium point E1 disappears and the economy converges to the axis equilibrium or to the zero equilibrium (the economy is extint).

More in detail: i)the equilibrium point P_1 is a stable node or focus as long $h < h_H = 0.31$ (i.e.); ii) at $h = h_H = 0.31$ the stable limit cycle appears, Fig. 1, 2, 3 report two-dimensional views respectively of: i) the monotonic convergence to the positive equilibrium, ii) of the involved cycle, and of the convergence to the axis equilibrium. The motion along the cycle is counterclockwise. The amplitude of this cycle increases, by increasing h up to[8] the point $h^{\pm} = 0.52$; beyond this threshold value of the replacement ratio and for the entire interval $h^{\pm} < h < 1$ the equilibrium P_1 is lost and the economy converges to the axis equilibrium and subsequently to the zero equilibrium .

Fig. 1. Monotonic convergence to P_1 at $h = 0{:}05$; I.C: w(0)=.40, E(0)=0.80

Fig.2. A stable limit cycle appeared at $h = h_H = 0{:}311$;I.C: w(0)=.40, E(0)=0.80

Fig. 3. Convergence to the axis equilibrium appeared at $h = h^{\pm} = 0{:}52$;I.C: w(0)=.40, E(0)=0.80

[8] The numerical simulations has revealed that when h further increases also an "explosion" of the system is possible if the wage adjiustment is very slow.

Table 2 reports in a synoptical view the process of phase transition in our model in terms of h.

Table 2. Windows of values of the replacement ratio h and related behaviour of the model

Windows of h	$(0; h_H)$	$(h_H; h^{\pm})$	$(h^{\pm}; 1)$
P_1	Stable (node or focus)	Stable limit cycle(and eventual 'explosion')	P_1disappears; possible extinct

Moreover the e¤ect of the replacement ratio parameter on the positive equilibrium of the state variables of the system is as expected from the analitycal results in the above section: when h increases income growth increases while the distributive wage share decreases: i.e when h_H =0.31 small ‡uctuations arise around a rate of total income growth of the 6.8%, and a wage share of 33%, when h = 0:05 the economy converges towards a rate of income growth of the 5.7% , and a wage share of 37%; in both cases the rate of employment (which is independent of h) remains constant at 83%.

For an illustrative example corresponding to the above numerical simulation, we recall that France (similar considerations can be made also for Spain), for which Rowthorn (1999, table 2) derives an elasticity of substitution of the CES function fairly represented by $\mu = 10$, has increased between the sixties and the nineties the replacement ratio from about 0.1 to about 0.5; this policy choice could have been responsible for 1) a lower (higher) long run rate of income growth, 2) the occurrence of ‡uctuations and of a possible instability, depending crucially on the level of childrearing costs of the workers and unemployed persons. In other words, if, as it is plausible, the cost of child rearing for unemployed persons is less than that for workers, policies increasing the replacement ratio when unemployment is higher in order to alleviate the higher social cost of unemployment could provoke, via e¤ects on the supply of labour, business and demographic cycles and ultimately either explosive oscillations or the transition towards an economically unplausible equilibrium.

6 Conclusions

So far the most part of the literature 1) has investigated the relation between unemployment and growth mostly within models of partial equilibrium and/or with exogenous growth; 2) has tackled the problem of the analysis of the relation between growth and unemployment in terms of either static or dynamic comparative exercise; and moreover 3) in order to study the ‡uctuations had to overimpose some exogenous stochastic shocks according to the Real Business Cycle theory (Stadler, 1990) We shall show in this paper that the Solovian growth with sluggishly adjusting, non market-clearing real wages and endogenous fertility can give rise to di¤erent dynamical outcomes, in particular to, loosely speaking, an extension of neoclassical growth with ‡uctuating employment rates, distributive shares, output and population.

Our paper answers simultaneously to two fondamental problems: 1) to explain the stylised fact of trend and cyclical ‡uctuations in the growth rate within an unique model and 2) do it in an endogenous as well as deterministic way. Moreover so far the literature has shown a negative relation between unemployment and growth (i.e. Bean-Pissarides (1993)), though it should be mentioned also the positive relation between unemployment and growth obtained in the particular "creative" disruption context according to the Schumpeter's idea (Aghion-Howitt, 1994). On the contrary our model shows very unexpected results on the role of the unemployment on growth (via e¤ects on population which is the engine of growth): either positive or negative relation unemployment-growth can be obtained according to the relative levels of cost of childrearing of workers and unemployed persons and the level of unemployment bene…ts.

We have argued that an increase of the unemployment bene…ts- as it has occurred in recent years in many countries as, for example, France and Spain - could work for the emergence of ‡uctuations and cause a positive e¤ect of the unemployment on the population and economic growth.

In particular, as we have shown that increasing replacement ratios when unemployment is higher in order to alleviate the higher social cost of unemployment could provoke, via e¤ects on the supply of labour, business and demographic cycles and ultimately either explosive oscillations or the transition towards an economically unplausible equilibrium, we have stressed a new role played from too high unemployment bene…ts [9].

Finally we indicate some possible extensions: a) from a macro point of view in order to have a full understanding of trend growth and cycles next steps should be 1) the introduction of monetary elements; 2) the introduction of an endogeneous unemployment bene…ts scheme, …nanced for instance either by workers'contributions or by income taxation; b) from a micro point of view to endogeneise the technical progress, noting that this latter can again depend on the population growth, for example through the introduction of a research sector as in the recent endogenous growth models (Jones, 1999).

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8 Appendix

8.1 Details on the derivation of the final model

The optimal factor demand ratio K/L can be obtained as solution of the cost minimization program by firms; from the first order conditions: (recall that $L_E = L\beta(t)$), (for simplicity we omit from now onwards the index time (t))

$$\frac{w}{r} = \frac{(1-z)\beta^{-\theta}}{z}\left(\frac{K}{L}\right)^{1+\theta} \tag{36}$$

from which the optimal factor demand ratio in terms of factor prices is obtained:

$$\frac{K}{L} = \left[\frac{zw}{(1-z)\beta^{-\theta}r}\right]^{\frac{1}{1+\theta}} \tag{37}$$

Now, by multiplying both members by $\frac{L}{K}$ the eq. (A.1)):

$$\frac{wL}{rK} = \frac{V}{(1-V)} = \frac{(1-z)\beta^{-\theta}}{z}\left[\frac{K}{L}\right]^{\theta} \tag{38}$$

and after some manipulation we obtain the optimal factor demand ratio in function of V. To obtain the labour productivity, we eleviamo alla (-1) both members of (38), 2) we add unity to both members, 3) we take account for the eq (A.3)

$$\frac{rK + wL}{wL} = \frac{1}{V} = \frac{zK^{-\theta} + (1-z)L_E^{-\theta}}{(1-z)L_E^{-\theta}} = \frac{\left[\frac{Y}{c}\right]^{-\theta}}{(1-z)L_E^{-\theta}} \tag{39}$$

from which it easily yields

$$\frac{Y}{L_E} = c\left[\frac{V}{1-z}\right]^{\frac{1}{\theta}} \Rightarrow \frac{Y}{L} = c\left[\frac{V}{1-z}\right]^{\frac{1}{\theta}}\beta \tag{40}$$

To obtain the product-capital ratio:

$$\frac{Y}{K} = \frac{Y L_E}{L_E K} = c\left[\frac{V}{1-z}\right]^{\frac{1}{\theta}}\left[\frac{(1-z)(1-V)}{zV}\right]^{\frac{1}{\theta}} = c\left[\frac{1-V}{z}\right]^{\frac{1}{\theta}} \tag{41}$$

The rate of profit is easily obtained:

$$\frac{P}{K} = \frac{YP}{KY} = c\left[\frac{V}{1-z}\right]^{-\frac{1}{\theta}}(1-V) = c(1-V)^{\frac{1+\theta}{\theta}}\frac{1}{z^{\theta}} \tag{42}$$

In the long run equilibrium (when the distributive share V is constant) the wage is growing to the following rate:

$$\frac{\dot{w}}{w} = \frac{\dot{\beta}}{\beta} \tag{43}$$

8.2 Windows of existence of P_1

The value h_* is determined from the condition $n(E_1) + a_0 > 0$ giving

$$(1 - a)\left[\frac{E_1}{q} + \frac{h(1 - E_1)}{q_u}\right] - \mu + a_0 > 0$$

i.e.

$$h > \frac{\frac{\mu - a_0}{1 - a} - \frac{E_1}{q}}{1 - E_1} q_u = h_*$$

The value h_{**} is determined from the condition $n(E_1) + a_0 < A$, giving

$$(1 - a)\left[\frac{E_1}{q} + \frac{h(1 - E_1)}{q_u}\right] - \mu + a_0 < A$$

i.e., after some manipulation

$$h < q_u \frac{A - a_0 + \mu - (1 - a)\frac{E_1}{q}}{(1 - a)(1 - E_1)} = h_{**}$$

8.3 The bifurcation condition

Let us consider the IFF condition for instability of the positive equilibrium $q(h_{**} - h_C) > F$; dividing bothe members by $q > 0$ we have.

$$h_{**} - h_C > \frac{q_u}{1 - a} \frac{\rho}{1 + \theta}$$

By introducing the expression for h_{**} given in the previous section, and putting $1/(1 + \theta) = K$ we obtain after some algebra

$$\frac{q(A - a_0 + \mu) - (1 - a)E_1 - (1 - a)(1 - E_1)}{q(1 - a)(1 - E_1)} > \frac{K\rho}{1 - a}$$

which in turn leads to

$$q(A - a_0 + \mu) > (1 - a) + K\rho q(1 - E_1)$$

Dividing by q one finally obtains the form presented in the main text

$$A - a_0 + \mu > \frac{(1 - a)}{q} + K\rho(1 - E_1)$$