SOLOW ON EXOGENOUS AND ENDOGENOUS GROWTH, AND THE SWAN PROPOSITION

by
Giacomo Costa
Dipartimento di Scienze Economiche
Università di Pisa
e-mail: costag@specon.unipi.it

0. Introduction.

In this note, we will review and discuss some of the ideas on exogenous and endogenous growth put forward by R. M. Solow in his Siena Lectures, [4].[1] On the one hand, a theory of endogenous growth would like to do with fewer exogenous parameters than the old, and this can be done only by broadening the scope of the theory itself, by providing an explanation of some phenomena that were in earlier theory left unexplained. The usual outcome of this process is that some parameters become variables, hence new interconnections arise. On the other hand, these new interconnections should manifest themselves in new properties of the steady state rate of growth: it should be greater than the natural rate of growth (the rate of growth of the effective labour force), and it should not be independent on the saving ratio. This seems to be Solow's conception of endogenous growth. In section 1, we review Solow's treatment of increasing returns to scale as a possible source of growth at more than the natural rate, and the associated proposition by Swan [7] about the general class of production function which are compatible with steady state growth. In section 2, we discuss some suggestions by Solow as to how to achieve endogenous growth, his "general point on endogenous growth models". Pursuing this, we explore first the connection, touched upon but not thoroughly discussed both by Solow and Burmeister and Dobell [3], between boundedness below of the marginal product of capital, the tendency of the competitive capital share to approach 1 when the growth rate of output and capital is higher than the natural rate of growth, the possible non-existence of a balanced growth equilibrium (where factors and product grow at a common proportional rate), and endogenous growth, meaning by this a type of growth where the dynamics of output is drawn by and ultimately converges to the dynamics of capital. In other words, we look for necessary an sufficient conditions under which endogenous growth obtains in the Solow model! This, however, as we will show, happens only if labour is an inessential factor of production and is therefore of limited interest, at least to people for whom the reproduction of labour is socially and institutionally different from the production of commodities. In section 3 the "transition" from exogenous to endogenous growth when labour is inessential is studied in greater detail in the context of a specific example. In section 4, we take up a tack that Solow considers "more promising", i.e., the endogenization of technical progress à la Arrow. Pursuing Solow's suggestion, we seek for the most general type of production functions, allowing learning of the Arrow type, which are compatible with steady state growth (where output and the reproducible factor grow at an equal, constant proportional rate). In section 5 we show that the resulting Arrow-Solow production function, although not at first sight of the Swan type, can be represented in that way. We offer an alternative representation of those increasing returns production functions that are compatible with steady states, and observe that while pursuing Solow's suggestion as to how to endogenize technical progress, we have as a side

---

[1] Robert Solow gave his Siena Lectures in 1992. I used his Lecture Notes for teaching the Siena students growth theory in the following years, and the present paper is in a sense a set of personal "Notes on Notes". In 2000, Solow integrated the material of his Siena lectures in the second edition of his book on growth theory, [6], with only few, if sometimes significant, adaptations.

product found the main elements for a new proof of the Swan Proposition. Section 6 is devoted to a scrutiny of Arrow's own production functions. The intriguing question of the admissible values of Arrow's learning parameter, raised by Solow in [5], is discussed and settled in the light of the Swan Proposition. In section 7, we briefly study the emergence of endogenous growth in the Arrow model, just for a unique real value of his learning parameter, and compare its trajectory to those generated by endogenous growth in the Solow model. In section 8, we pose, and to some extent answer, the problem of finding a Swan-like proposition for endogenous growth. We conclude in section 9 with a summary of our observations on Solow's "general point", and some comments of our own on exogenous and endogenous growth.

1. Increasing returns and the Swan proposition.

The basic growth model is made up of the two equations

(1) $Y = F(K, N, t)$,

(2) $dK/dt = sY$,

where the production function F is first degree homogeneous in its first two arguments (in the economic jargon, it is "constant returns to scale"), Y is output, K is capital, N is the labour force, and t is the time index. All variables are functions of time, in particular, N depends on t only and is exponential. It is usually assumed that $F(0, 0, t) = 0$ for all t, and we will do so in the following. The parameter s is the average saving propensity. The system is completed by specifying initial conditions for the factors of production, K and N.
Solow is explicit in rejecting the view that the essence of endogenous growth is increasing returns to scale.[2] Starting with a production function

(1') $Y = F[K, (AN)^h]$,

first degree homogeneous in his two arguments, and $h > 1$, so that F is increasing returns in K and N, and A is exponential function of time with exponent a, Solow finds that the steady state rate of growth, g, exists and is

(*) $g = (a + n)h > (a+n)$,

where we notice that $(a+n)$ is the "natural" rate of growth. He observes [4, p. 8] that "if there are increasing returns in the only form that allows for exponential steady state, then you get a rate of growth that is exogenous". The rate of growth given by (*) is exogenous because it is independent of s. There is another, more cryptic side of this comment that should be explained. Solow is alluding to the circumstance, probably a piece of folklore among growth theorists,[3] that *the class of functions (1'), with F constant returns in its two*

---

[2] The production function (1) exhibits <u>increasing returns to scale</u> iff $F(\lambda K, \lambda N, t) > \lambda F(K, N, t)$ for $\lambda > 1$, all semi-positive (K, N), and all t. Clearly this condition holds if and only iff $F(\lambda K, \lambda N, t) < \lambda F(K, N, t)$ for $0 < \lambda < 1$, all semi-positive (K, N), and all t.

[3] Notice however that it is not reported in such a thorough, competent graduate textbook in growth theory as Burmeister and Dobell [3]. Indeed, from their exposition on p. 35 one might draw the impression that for all types of increasing (and decreasing) returns there is a steady state rate of growth, and an associated K/AN equilibrium ratio. Solow mentions again, although no less cryptically than in [4], the Swan proposition in [5, p. 15] in a passage where he is comparing the old and the new theory of growth: "Is the assumption of constant returns to scale the same sort of non robust assumption, only in this case the one that pervades the Old Growth Theory? Constant returns to scale is certainly a borderline case, unlikely to turn up in practice.

arguments, and h positive, exhausts the functions compatible with steady state growth. Notice that this result answers at once to the two questions of (i) what sort of technical progress is compatible with a steady state, and (ii) how the function (1) is to be specified to allow for increasing or decreasing returns to scale, in a form that should be compatible with steady state growth. Since the proposition was, to my knowledge, first introduced by T. Swan in [7], I shall call it "Swan's Proposition".[4] Thus by invoking Swan's Proposition Solow argues that a formula like (*) is possible if and only if the production function (1) is of type (1'). Having shown that the increasing returns assumption is not by itself the key to endogenous growth, Solow in his discussion proceeds to set $h = 1$ in (1'), thus obtaining

(1'') $Y = F(K, AN)$,

the standard production function of the old neo-classical theory.

2. Some equivalent conditions for endogenous growth in Solow's own model

In [4, pp. 39-41], Solow makes a "general point on endogenous growth models" that should help to establish a bridge between the "old" an the "new" growth theories, and that we wish to review and if possible to develop.

By differentiating with respect to time the production function (1''), Solow obtains

(3) $dY/Ydt \equiv g = [KF_K /Y]\, dK/Kdt + [ALF_{AN}/Y](a+n)$,

and inserting the saving function (2)

$$dY/Ydt \equiv g = sF_K + [ALF_{AN}/Y](a+n).$$

Letting the elasticity of output with respect to capital be β,

(4) $\beta \equiv KF_K/Y$,

so that, by constant returns to scale,

(4') $ANF_{AN}/Y = 1- \beta$,

Solow finds

(5) $g = sF_K + (1-\beta)(a+n)$,

a formula which can be rearranged to show the difference between the actual and the natural rates of growth,

(5') $g - (a+n) = sF_K - \beta(a+n)$,

The difference is that Old Growth Theory can live comfortably with increasing or decreasing returns to scale. It is true that if we insist on generating exponential steady states, non-constant returns have to enter in a certain special way; but that is purely a matter of convenience and does not affect fundamentals."

[4] Swan's own statement [7, p. 208] is: "If our economy is to be structurally capable of enjoying a golden age the function $Y = f(K, N, t)$ must take the form $Y = f(K, N^q e^{mt})$, where f is homogeneous of degree one in K and $N^q e^{mt}$." Since in this formulation only N (rather than the whole product AN as in (1)) is raised to the power q, the steady state rate of growth turns out to be $g = qn + m$. The two representations are equivalent. In the very interesting Appendix to his paper, Swan offers a proof of his proposition.

a formula that neatly expresses, according to Solow [3, p. 40], the challenge that any theory of endogenous growth must meet: "If output only grows at a rate equal to $(a+n)$, then this a model of exogenous growth. There is endogenous growth only when the left hand side, and therefore the r.h.s., is positive. Generally what makes the r.h.s. not to be positive is that F_K falls as capital accumulates. Thus, we can say that the job of any model of endogenous growth is simply to keep the marginal product of capital from falling too fast as capital accumulates."

Now this argument might appear a little too cursory, since F_K and β are not independent of each other but rather related by (4). However, if we rewrite (5'), taking into account (4), as

$$(5'') \qquad g - (a+n) = (KF_K/Y)[sY/K - (a+n)] ,$$

and, since in steady state $g = dK/Kdt$, as

$$g - (a+n) = (KF_K/Y)[g - (a+n)] = \beta[g-(a+n)].$$

we see that $g - (a+n)$ can be positive only if $\beta = 1$; since in s.s., K/Y is a positive constant, $\beta = 1$ only if $F_K > 0$. Thus, Solow's remark might be right after all. Very few c.r.s. production functions, however, allow for β to actually achieve the value 1.[5] Some do not even allow β to tend to 1 as AN/K approaches zero, for example, the Cobb-Douglas production function. Some, however, like the C.E.S., do. In the following, we will look for a characterization of the most general class functions for which $\lim_{(AN/K)\to 0} \beta = 1$. Before taking this up for consideration, it is important to understand exactly what is at issue: the emergence of endogenous growth within the old Solow model itself. Let us rewrite (3) as

$$dY/Ydt \equiv g = [KF_K /Y]\, dK/Kdt + [ALF_{AN}/Y](a+n) = \beta dK/Kdt + (1-\beta)(a+n).$$

It is a key feature of the neo-classical growth model that the balanced state rate of growth can be found without using the saving function. In steady state, K/Y is constant by definition hence the proportional rates of growth of output and capital are equal. Since in general, for all positive finite AN, K, we have $\beta < 1$, we can solve the equation

$$g = \beta g + (1-\beta)(a+n)$$

to find

$$g = (a+n),$$

and then use the saving function (2)

$$dK/dt = (a+n)\, K = sY$$

to find the steady state capital-output ratio

---

[5] For example, the production function $F(K, AN) = uK$, where u is a positive constant, or, more generally, $F(K, AN) = \mathrm{Min}(uK, vAN)$, where u and v are positive constants, and $K \leq (v/u)AN$. The condition is also satisfied by production functions of the separable class $F(K, AN) = Kf(AN)$, but these are not, in general, c.r.s. See section 6 for a discussion of one important example of this type.

(6) $(K/Y) = s/(a+n)$,

and the steady state factor proportion *AN*/*K* from the production function (1'')

(7) $Y/K = F(1, AN/K)$.

It may happen, however, that there is no positive (or even non-negative) solution for *AN*/*K* to the equation

$$(a+n)/s = F(1, AN/K)$$

In other words, the range of values of the increasing function $Y/K = \varphi(AN/K)$ implicitly defined by (7) may be bounded below,[6]

(7') $\varphi(AN/K) \geq (a+n)/s$ for all positive AN/K.

By continuity, we will also have

(7'') $\varphi(0) \geq (a+n)/s$.

The average productivity of capital may remain too high. Let's consider the implications of (7'). By (5''), we see that, whatever the initial *AN*/*K* ratio may be, there will be growth both of capital and of output at more than the natural rate. But will there be an at least asymptotic steady state? Only if $\beta \to 1$ as $AN/K \to 0$. Will this be necessarily the case? By (7''),

$$\varphi(0) = F(1, 0) \geq (a+n)/s > 0,$$

hence labour must be an <u>inessential</u> factor of production.[7] Let's state this as our

Lemma 1: if there is no positive *AN*/*K* balanced growth ratio, then labour is an inessential factor of production.

By c.r.s., inessentiality of labour implies

$$F(K, 0) = KF(1, 0) \equiv bK > 0 \quad \text{for all positive K},$$

and therefore

$$F(1, 0) = F_K(1, 0) = b > 0.$$

[6] We will of course assume, for the whole of the present section, that there are values of *AN*/*K* for which the inequality (7') is satisfied.
[7] See Burmeister and Dobell [3, p. 30] for a definition of this concept. Burmeister and Dobell provide a good discussion of the connection between non-essentiality of labour and non-existence of a steady state solution to the neo-classical growth model. They go so far as observing (p. 34) that "the more flexible production conditions we have introduced, relaxing the assumption that both factors are essential, do introduce the logical possibility that this simple economy may be unable to attain a balanced growth equilibrium, and hence also the logical possibility that shifts in saving behaviour may affect the long-run growth rate".

On the other hand, (4) gives

$$\beta \equiv KF_K/Y = KF_K/(Y/K) = F_K/F(1, AN/K),$$

so that

(8) $$\lim_{(AN/K)\to 0} \beta = \lim_{(AN/K)\to 0} F_K / \lim_{(AN/K)\to 0} F(1, AN/K) = b/b = 1.$$

We have therefore established the following

Lemma 2: if labour is an inessential factor of production, the competitive capital share tends to 1 as the ratio AN/K tends to zero,

and also, combining the two lemmas, the following proposition, which gives a sufficient condition for endogenous growth:

Proposition 1: if there is no balanced growth equilibrium, there is an asymptotic steady state, where the competitive capital share absorbs the whole output and capital and output grow at a rate, higher than the natural rate of growth.

Along the time path of (endogenous!) growth, the (K/Y) ratio is increasing (since $\varphi(AN/K)$ is increasing in its argument), but it converges to the finite value b. The rates of growth of capital, dK/Kdt, and output, g, are decreasing, $dK/Kdt > g$ for all t, while their common limiting value is the Harrod-Domar "warranted rate", sb.

Conversely, suppose labour is an essential factor of production,

$$Y = F(K, 0) = 0 \text{ for all non-negative K.}$$

By c.r.s.,

(9) $$Y = F(K, 0) = KF(1, 0) = 0 \text{ for all } K \geq 0,$$

By continuity of the production function,

$$\lim_{(AN/K)\to 0} (Y/K) = \lim_{(AN/K)\to 0} F(1, AN/K) = 0,$$

so that

(10) $$\lim_{(AN/K)\to 0} \varphi(AN/K) = 0 .$$

We can restate (10) in the following

Lemma 3: essentiality of labour implies the existence of a positive balanced growth AN/K ratio.

What does essentiality of labour imply for the behaviour of β ? By the zero degree homogeneity of the F_K function, we have

(11) $F_K(K, AN) = K^0 F_K(1, AN/K) = F_K(1, AN/K)$,

and by the first degree homogeneity of the *F* function,

$$(Y/K) = F(1, AN/K).$$

We can then express β as

(12) $\beta = F_K(1, AN/K)/F(1, AN/K) = [F(1, AN/K) - (AN/K)F_{AN}(1, AN/K)]/F(1, AN/K) =$

$$= 1 - [(AN/K)F_{AN}(1, AN/K)]/F(1, AN/K) = 1 - R,$$

where by (9),

$$R = F_{AN}(1, AN/K)/ [(F(1, AN/K) - F(1, 0))/(AN/K)].$$

Now,

$\lim_{(AN/K)\to 0} R =$

$$= \lim_{(AN/K)\to 0} F_{AN}(1, AN/K)/ \lim_{(AN/K)\to 0} [(F(1, AN/K) - F(1, 0))/(AN/K)],$$

which is equal to 1 if $\lim_{(AN/K)\to 0} F_{AN}(1, AN/K)$ is finite. Thus by (12)

(13) $\lim_{(AN/K)\to 0} \beta = 1 - \lim_{(AN/K)\to 0} R = 0$ when $\lim_{(AN/K)\to 0} F_{AN}(1, AN/K)$ is finite.

However, as the example provided by the Cobb-Douglas production function illustrates, $\lim_{(AN/K)\to 0} F_{AN}(1, AN/K)$ may well be infinite, and $\lim_{(AN/K)\to 0} \beta$ may take up a value in the open interval (0, 1). However, we can establish that

(14) $\lim_{(AN/K)\to 0} R = 0$,

and therefore $\lim_{(AN/K)\to 0} \beta = 1$, is impossible.
Suppose indeed that (14) were true. Then for all $\varepsilon > 0$ there would be η such that for all $AN/K < \eta$,

(15) $(AN/K)F_{AN}(1, AN/K) < \varepsilon F(1, AN/K)$.

However,

(16) $F(1, AN/K) = F(1, AN/K) - F(1, 0) = F_{AN} (1, (AN/K)_o)AN/K < F_{AN} (1, AN/K)AN/K$,

where the second equality, with $0 < (AN/K)_o < AN/K$, is the mean value theorem, and the inequality is due to the fact that *F* is increasing in its second argument. Composing (15) and (16), we find

$$1 < \varepsilon,$$

a contradiction.

Thus, in the light of Lemma 2, we can state the following lemma, which gives a complete answer to our problem as to the class of production functions for which $\lim_{(AN/K)\to 0} \beta = 1$:

Lemma 4: $\lim_{(AN/K)\to 0} \beta = 1$ if and only if labour is inessential.

Thus if labour is essential, by Lemma 3 there is certainly a balanced growth equilibrium *AN*/*K* ratio, and, by Lemma 4, there can be no asymptotic steady state where output grows at a higher than the natural rate, i.e., endogenous growth. We can then conclude with a proposition ad a corollary:

Proposition 2: In the context of the Solow model, there is an asymptotic steady state, where the competitive capital share absorbs the whole output and capital and output grow at a rate, higher than the natural rate of growth, if and only if there is no positive balanced growth *AN*/*K* ratio.

Corollary: In the context of the Solow model there can be endogenous growth only if labour is inessential.

3. Endogenous growth under constant returns to scale and non-essentiality of labour: a worked out example

It may be interesting to see how this rather abstract discussion applies in practice. Let's take up Solow's example,[8] indeed, the special case of it given by the c.r.s. production function with inessential labour

(1S) $$Y = F(K, AN) = BK + K^b(AN)^{1-b},\ B > 0,\ 0 < b < 1,$$

with associated average and marginal productivity of capital

(1'S) $$(Y/K) = \varphi(AN/K) = B + (AN/K)^{1-b},$$

(1''S) $$F_K = B + b(AN/K)^{1-b},$$

both bounded from below by *B*. The growh equation (3) becomes now

(3S) $$dY/Ydt \equiv g = \beta dK/Kdt + (1-\beta)(a+n),$$

where

$$\beta = [B+b(AN/K)^{1-b}]/[B+(AN/K)^{1-b}],$$

and we notice that $\beta < 1$, but $\beta \to 1$ as $(AN/K) \to 0$. Now we see from (6) and (1'S) that a necessary condition for the existence of a steady state rate of growth is that

(4S) $$(Y/K) = B + (AN/K)^{1-b} = (a+n)/s,$$

[8] See Solow [4, pp.30-31, 40-41], and his somewhat more detailed exposition in [6, pp.138-140, pp.144-147]. In neither work does he make the connection between this brand of endogenous growth and the possible non existence of a balanced growth equilibrium in his own model. Burmeister and Dobell [3, pp. 31-35] do notice and discuss it to some extent.

and since in a balanced growth equilibrium (AN/K) must be positive,

(5S) (a+n)/s > B, (a+n) > sB.

Suppose that

(6S) dK/Kdt = sY/K > (a+n).

Then (AN/K) $\rightarrow 0$ as $t \rightarrow \infty$, hence $\beta \rightarrow 1$ and $g = g(t) \rightarrow dK/Kdt$. Since from (1'S) the positive number B is a lower bound for (Y/K), we can see from (4S) that a sufficient condition for (6S) to hold is that

(7S) $sB \geq (a+n)$,

precisely the negation of (5S), a necessary condition for the existence of a balanced growth equilibrium. Under (6S), $dK/Kdt = sY/K = s(B + (AN/K)^{1-b})$ is a decreasing function of K/AN and of time, and converges monotonically from above to B. From (3S), the same is true of $g(t)$. Notice in particular that

$$\lim_{t\rightarrow\infty} g(t) = (a+n) \text{ if } sB = (a+n),$$

thus, in this borderline case in which there begins to be no steady state rate of growth, there is still an asymptotic proportional rate of growth.

The development of the preceding and this section is suggestive. One of the implications of Proposition 1 is that by choosing a sufficiently high rate of saving, an economy can free itself from the shackles of the natural rate of growth and grow at a "warranted rate" of its choice. Moreover, as is made clear from (3'), it can grow at its chosen warranted rate of growth even if the natural rate of growth is zero! In a sense, this is however not surprising. By our Corollary, a necessary condition for endogenous growth is that labour be an inessential factor of production. This is why the reproducible factor, K, is able to grow faster than ($a+n$), and to draw the growth of output. As Solow argues, "there are models in the literature that proceed in exactly this way." These models are just special cases of his own, old model. However, "Another more interesting way to proceed, …is genuinely to endogenize technological progress "

4. Endogenizing technical progress à la Arrow

"The first paper in this tradition is of course Arrow's paper on learning by doing," Solow continues. "What one could do is to suppose that that the level of technology depends on the amount of capital that has been accumulated." And he suggests the formulation

$$Y = F(K, A(K)N),$$

but I think that the function

(1*) $Y = F(K, A^*(t, K)N)$,

that allows both an exogenous and an endogenous component in the labour productivity function A* is more in the spirit of his argument.[9] An argument that we want to pursue, since curiously Solow did not do so himself. [10]
By differentiation of (1*) with respect to time, and some rearrangements, we find

$$(17)\qquad g = [\beta + (1-\beta)\eta](dK/Kdt) + (1-\beta)[(A^*_t/A^*) + n],$$

where η is the elasticity of A^* with respect to K, i.e.,

$$(18)\qquad \eta \equiv (KA^*_K/A^*).$$

Now, if we are to have a "natural rate of growth" in this model, with which to compare the actual rate of growth g, A^*_t/A^* must be independent of t and K, hence must equal some positive constant, a. Given this, if there is to be some g independent of t and of K, η must itself be independent of t and K. Both these conditions are satisfied if and only if :

(i) the function $A^*(t, K)$ has the following separability property:

$$A^*(t, K) = A(t)B(K),$$

which is necessary for A^*_t/A^* to be independent of K and η to be independent of t;

(ii) $\qquad A(t) = \exp(at),$

necessary to make $A^*_t/A^* = A_t/A$ independent of time, and

(ii') $\qquad B(K) = K^b$, b = some positive constant,

necessary to make η independent of K. Notice that in particular, by (18), $\eta = b$.

Under these conditions, (17) can be rewritten as

$$g = [\beta + (1-\beta)b](dK/Kdt) + (1-\beta)(a + n),$$

and we see that there will be a steady state rate of growth if the equation

$$g = [\beta + (1-\beta)b]g + (1-\beta)(a + n)$$

has a positive solution. Clearly, this requires $b < 1$. Existence of the solution implies in this case its uniqueness,

$$(**)\qquad g = (a+n)/(1-b),$$

---

[9] In the sequel, we will denote by A^*_t and A^*_K the partial derivatives of the A^* function with respect to its two arguments.

[10] He did not pursue it in his Siena Lectures. However, in a sense a few years later he devoted his whole booklet [5] to pursuing it. And he did come out with the right answer, of course. It is too bad he did not fully integrate his findings on Arrow in his later book on growth theory [6]. In [4, p.41], he had written: "It is perfectly possible for the quantity $A(K)$ to grow fast enough as capital accumulates to keep the partial derivative of F with respect to the first argument from going to zero". And in [6, p. 147], he adds: "The details can be left as an exercise; the economically interesting thing is the story told about $A(K)$." Thus we are doing the exercise he left for his readers. The answer is Proposition 3.

For $b = 1$, the rate of growth of output is infinite; for $b > 1$, the increasing returns generated by "learning by doing" are so strong than only if effective labour grew at a negative rate could there be a steady state solution. Formula (**) is completely analogous to Swan's (*) of section 1. No dependence of g on β, a parameter that summarizes the properties of the production function F; no dependence on the saving propensity, s, either, hence, by Solow's own suggested definitions, even if $g > (a+n)$, we are still in the realm of exogenous growth. Notice that if $(a+n) = 0$, $g = 0$. Only if the natural rate of growth is positive can there be a positive (if bigger) s.s. rate of growth. This too can be considered as a property of exogenous growth. Endogenous growth, therefore, should contemplate $(a+n) = 0$ and $g > 0$.

5. Equivalent representations of the production function in the Swan proposition

By following Solow's suggestion as to how to achieve endogenous growth we have established the following

Proposition 3: a production function of type (1'') is compatible with steady state growth if and only if the function A^* takes the form

$$A^*(t, K) = K^b A, \quad 0 < b < 1,$$

where A is the usual exponential function.

It is easy to check that the resulting production function

$$(1^{**}) \qquad Y = F(K, K^b AN), \quad 0 < b < 1$$

exhibits increasing returns in the variables K and AN. It does not at first sight look of the class (1') indicated by the Swan Proposition. Is this then a class of functions that has somehow managed to slip through the mesh of the Swan Proposition? Of course not. Notice that

$$(19) \qquad F(K, K^b AN) = KF(1, (K/AN)^{b-1} (AN)^b),$$

where the function

$$G(K, AN) = KF(1, (K/AN)^{b-1})$$

is c.r.s. Now,

$$(20) \qquad G(K, (AN)^q) = KF(1, (K/AN^q)^{b-1}) = KF(1, (K/AN)^{b-1} (AN)^{-(q-1)(b-1)}),$$

and we see that the l.h.s. of (19) and of (20) are equal for the value of q that satisfies the equation

$$b = -(q-1)(b-1),$$

i.e. for

$$q = 1/(1-b).$$

Suppose instead we were given a function $G(K, (AN)^q)$, with G c.r.s. in its two arguments and $q > 0$.

(21) $$G(K, (AN)^q) = KG(1, (AN)^q/K) = KG(1, (AN/K)^q K^{-(1-q)}),$$

and we notice that the function

$$H(K, AN) = KG(1, (AN/K)^q)$$

is c.r.s. Now,

(22) $$H(K, K^b AN) = KG(1, (K^b AN/K)^q) = KG(1, (AN/K)^q K^{bq}),$$

and we see that the equality between the l.h.s. of (21) and (22) is achieved for that value of q which solves

$$bq = -(1-q),$$

$$b = (q-1)/q > 0.$$

We can therefore conclude with the following "equivalent representation proposition":

Proposition 4: A production function can be expressed in the form (1**) if and only if there exists a c.r.s. function G and a value of the parameter q, $q = 1/(1-b) > 1$, such that

$$F(K, K^b AN) = G(K, (AN)^q).$$

We are now in the position to review and assess the second part of Solow's "general point". If F is first degree homogenous in its two arguments, F_K is zero degree homogenous, i.e., depends on the ratio K/AN only. F_K can only be constant over time, therefore, if K growths at the same proportional rate as AN, and therefore, if Y growths at the natural rate $(a+n)$. Increasing returns of the form (1**) do not, as we have seen, disturb the c.r.s. property of the F function with respect to its two arguments, K and $K^b AN$. It follows that F_K will be constant over time if the ratio $K/K^b AN$ can be kept constant, or in other words, if g satisfies equality (**). This is the arithmetical way in which Solow's suggested route "to keep the marginal product of capital from falling too fast as capital accumulates" works in the model with endogenous technical progress. "The job of any model of endogenous growth" has been carried out by a model of exogenous growth, with the right type of increasing returns.

It may be of interest to remark how close we have come to give a proof of the Swan Proposition. The proof offered by Swan in [7] is not long, but relies on a rather deep theorem on partial differential equations that most economists would prefer to bypass if they could. An alternative proof of the theorem can be proposed, exploiting some of the results contained in the present paper. Necessity: (i) the observation that the only type of technical progress compatible with s.s. growth is labour increasing, $F(K, N, t) = F(K, AN)$;[11] (ii) the observation that the only way in which A^* may depend on K is exponential, $F(K, A^*(t, K)) = F(K, K^b AN)$: this our Proposition 3 ; (iii) the alternative representation

[11] See [4, pp. 5-7], or [6, pp. 110-111], for a very elegant proof of this result, a "mini-Swan Proposition".

proposition, our Proposition 4. Sufficiency: logarithmic differentiation of the production function (1’) , and use of the s.s. condition K/Y constant over time.

6. The Arrow production functions and the Swan proposition

Let’s consider Arrow’s own production function

(1A) $Y = aK[1-(1-(N/cK^{1-n}))^{1/(1-n)}] = F(K, N).$

As Arrow points out [1, p. 159], $F(K, N)$ has increasing returns to scale for all positive real values of n, and since $F(K, 0) = 0$ for all K, labour is certainly essential. We notice that its two variants

(1A’) $Y = aK[1-(1-(N^{1-n} /cK^{1-n}))^{1/(1-n)}] = F^1(K, N)$

and

(1A’’) $Y = aK[1-(1-(N/cK))^{1/(1-n)}] = F^2(K, N)$

are both constant returns to scale functions. Also,

$$F^1(K, N^q) = F(K, N) \text{ , with } q = 1/(1-n) \text{ ,}$$

and

$$F^2(K, K^bN) = F(K, N), \text{ with } b = 1-n.$$

We notice that we have managed to express the Arrow’s production function (1A) in either of the forms indicated in our “alternative representation” proposition. For $0 < b <1$ then, and only for b in this interval, the Arrow model falls within the Swan Proposition. We are now ready to make the following

Proposition 5: by the Swan Proposition (and if necessary Proposition 4), only if $n < 1$ does the production function (1A) allow for a proportional rate of growth of the same sign as the natural rate of growth.

Arrow [1] seems to claim that his model would have a steady state solution for all positive real values of n. In his highly interesting lectures on the Arrow growth model Solow [5, pp.7-13] gives several instructive euristic arguments to rule out $n \geq 1$. The above analysis show quite well what happens: as n goes through 1, the rate of growth goes through infinity, and comes back to… reality, so to speak (to the real numbers), but with sign opposite to that of the natural rate of growth.[12]

7. Endogenous growth under the Arrow exception

Let us now consider the production function

(2A) $Y = aK(1- e^{-N/b}),$

[12] It can be checked that the β associated to (1A) is less than 1 only for $n < 1$. But the advantage of the Swan Proposition is that by applying it one does not have to undertake the computations of β!

where a and b are some positive constants. (2A) can be obtained from by letting $n \rightarrow 1$ in (1A), or from the more general production set up by Arrow. We can express this production function as

(2A') $Y = F(K, N) = aKf(N)$,

where

(3A) $f(0) = 0,\ 0 < f(N) < 1,\ \lim_{N\rightarrow\infty} f(N) = 1,\ f'(N) = (1/b)e^{-N/b} > 0$,

and

(4A) $F_K = af(N),\ F_{KK} = 0,\ F_N = aKf'(N) = (aK/b)e^{-N/b},\ F_{NN} = -(aK/b^2)e^{-N/b} < 0$,

(5A) $\alpha \equiv NF_N/Y = Nf'(N)/f(N) \equiv \alpha(N) = (N/b)e^{-N/b}/[(1- e^{-N/b})] > 0$,

(6A) $\beta = KF_K/Y = 1$.

In particular, it follows from (5A) that

(7A) $\lim_{N\rightarrow\infty} \alpha(N) = 0$.

We can see from (3A) that the production function (2A) is increasing returns to scale, and under it labour is essential. As we have already remarked, it does not belong to the Swan class. A steady state at a positive constant proportional rate of growth is therefore impossible. Or rather, it is impossible, given that $N(t) = N_o e^{nt}$, if $n > 0$.[13] By differentiating (2A') with respect to time, we find

(8A) $dY/Ydt = g(t) = \beta dK/Kdt + \alpha(N).n = sY/K + \alpha(N).n = saf(N) + \alpha(N).n$,

where we note that we have found in passing

(9A) $dK/Kdt = saf(N)$,

so that by (3A)

(10A) $\lim_{t\rightarrow\infty} (dK/Kdt) = sa$.

Along the endogenous growth trajectory, the rate of growth of capital increases at a decreasing rate, and approaches the warranted rate of growth sa asymptotically. (8A) and (7a) give

(11A) $g(t) > dK/Kdt$ for all t,

and

(12A) $\lim_{t\rightarrow\infty} g(t) = sa$.

[13] Notice that here n is again the rate of growth of the working population, not the learning parameter of the Arrow production model discussed in the previous section.

Thus in the special case $b = 1$ the Arrow growth models yields endogenous growth, as Solow remarks in [4, pp. 11-12]. Although we have concentrated on Arrow's special case, it should be clear that the same conclusion holds for all production functions $F(K, N) = aKf(N)$ where $f(N)$ satisfies

(3A') $\quad f(0) = 0;\ 0 \le f(N) < U,\ U > 0;\ f'(N) > 0.$

To see the importance of the second condition (3A'), consider the production function

(1B) $\quad Y = KN^{\alpha},\ 0 < \alpha < 1,$

currently highly popular, it would appear, among the new growth theorists.[14] Only if $n = 0$ can the endogenous rate of growth $g = sN^{\alpha}$ be obtained. The production functions (3A') give a steady state rate of growth $g = saf(N)$ if $n = 0$, an asymptotic steady state if $n > 0$. This is the content of the following

Proposition 6: All $F(K, N) = aKf(N)$, where $f(N)$ satisfies (3A'), generate endogenous growth.

The similarity between the growth paths discussed in sections 3-4 and in the present is striking. In both cases the rate of growth of output converges to the rate of growth of capital as the latter converges monotonically to its long run Harrod-Domar value. However, in the model of sections 3-4 the rate of growth of capital is decreasing, while in the present it is increasing with time. In the former the rate of growth of output is smaller, in the latter larger, than the rate of growth of capital. In the former the capital-output ratio is decreasing, in the latter increasing, with time. Only in the latter we have the possibility of a steady state growth at a constant positive level with a fixed working population. It is certainly intellectually satisfying to establish that an economy with a non-reproducible factor of production in fixed amount can grow at a positive constant proportional rate. All previous growth theories, classical, neo-classical, or others, had shied away from this. The separation of the actual from the "natural rate of growth" is now complete. What is more, for the class of production function mentioned in Proposition 6 labour is an essential factor of production. It can be argued, however, that the class of production functions (3A') is exceptional.

8. Toward a Swan-like proposition for endogenous growth

A Swan-like proposition for endogenous growth would answer the following question, a question that may have already occurred to the reader: what is the most general type of production function allowing for essential labour and endogenous growth? Our only example so far has been the class of production functions mentioned in Proposition 6. As is clear from (6A), (3A'), all production functions of this class, because of their very special separable structure, have $\beta = 1$. Can't we have increasing returns and $\beta \to 1$?

Consider the following production function

[14] See Bertola [2, p.1187]. The new growth theorists claim that they do not assume (1B), but that they "derive" it. How? They start with the innocent looking $Y = A(t)K^{1-\alpha}N^{\alpha}$, introduce an Arrow equation for $A(t)$, $A(t) = K^{b}$ as in section 4 above, to find $Y = K^{b}\,K^{1-\alpha}N^{\alpha}$. By some miracle, $b = \alpha$, an so (1B) is obtained. One can sympathize with Solow's dismay at similar procedures.

(23) $Y = F(K, AN) = cK[1-\exp(-(AN)^p/K)]$,

where c is a positive constant and $p \geq 1$ is a parameter. If $p = 1$, (23) is c.r.s. Let's assume $p > 1$. Then (23) is increasing returns to scale: for a proportional increase in K and AN, the term $((AN)^p/K)$ increases, the term $\exp(-(AN)^p/K)$ decreases, the whole expression within square brackets therefore increases. Thus, the increase in Y is more than proportional to the increase in K and AN that produces it. Some easy computations give

(24) $F_K = c[1 - \exp(-(AN)^p/K)(1+(AN)^p/K))]$,

(25) $\beta = [1 - \exp(-(AN)^p/K))(1+(AN)^p/K)]/[1 - \exp(-(AN)^p/K)]$,

(26) $ANF_{AN}/Y \equiv \alpha = p((AN)^p/K) \exp(-(AN)^p/K)/[1- \exp(-(AN)^p/K)]$.

It can be seen from (25) that $0 < \beta < 1$ and that, if $(AN)^p/K \to\infty$, then $\beta\to 1$; from (26) that $\alpha > 0$ and $\alpha\to 0$ as $(AN)^p/K \to\infty$. The equation for the rate of growth of output becomes

(27) $g(t) = \beta dK/Kdt + \alpha(a+n) = sF_K + \alpha(a+n) =$

using (24)

$$= sc[1 - \exp(-(AN)^p/K)(1+(AN)^p/K))] + \alpha(a+n).$$

Suppose $\lim_{t\to\infty} ((AN)^p/K) = \infty$. Then $\lim_{t\to\infty} g(t) = sc$, the asymptotic rate of growth of output is the warranted rate of growth. Now, notice that by (23)

(28) $dK/Kdt = sc[1-\exp(-(AN)^p/K)] < sc$ for all AN, K,

so if

(29) $p(a+n) > sc$,

the expression $((AN)^p/K)$ will indeed tend to infinity and there will be endogenous growth. Thus (29) is a sufficient condition for endogenous growth. If in addition $sc > (a+n)$, so that altogether

(30) $(a+n) < sc < p(a+n)$,

the rate of growth of output gradually rises and approaches the "warranted rate of growth" sc in the limit. By (23) the output-capital ration is an increasing function of time, so $dK/Kdt < g(t)$ for all t, although of course by (28) $\lim_{t\to\infty} (dK/Kdt) = \lim_{t\to\infty} g(t) = sc$.

Proposition 7: the class production functions (23), parametrized by p, allows for endogenous growth under condition (30).

This example is interesting not only because it consists of a one parameter family of production functions rather than just a single, isolated function on the brink of non-existence like (2A), but also because the emancipation of the warranted from the natural rate of growth is here incomplete: for (29) requires $(a+n) > 0$. No such requirement is needed for endogenous growth when the production function is

(31) $Y = F(K, AN) = cK[1- \exp(ANK)]$,

an example that is worth mentioning because the α and β associated to it depend on *ANK* rather that *K*/*AN*, and reach their limiting values 0 and 1 for $(ANK)\rightarrow \infty$, a condition that is implied by $(AN/K) \rightarrow 0$ since at the least $(a+n) = 0$. Moreover, the β associated to (31) approaches 1 from the right,[15] a possibility that under c.r.s. cannot arise.

The class of production functions with essential labour which are compatible with endogenous growth has some features that by now should be clear: as $AN/K\rightarrow 0$, β should tend to 1, α to 0. As a consequence, $(\alpha+\beta)$ tends to 1: the increasing returns production function should gradually but irresistibly turn into a constant returns to scale function of the highly simplistic form $Y = cK$. It has not been established above that a functional form of the separable type $Y = F(K,AN) = KG(K, AN)$, with $G(K, 0) = 0$, like (23), is necessarily implied by these conditions. To this extent, the "Swan problem for endogenous growth" remains open.[16]

9. Conclusions.

As we have seen, Solow's "general point about endogenous growth models", although in no sense incorrect, is not as sharp and enlightening as one might have thought. Our attempts at elucidating it in several directions can be summarized as follows. In the context of the old neo-classical theory, we can get endogenous growth with $\beta = 1$, but, if we insist on maintaining one of is main ingredients, constant returns to scale, we must resign ourselves to very simplistic production functions; or we can settle for $\beta\rightarrow 1$, which opens up for endogenous growth a broad field: a circumstance that surprisingly Solow himself has chosen to let pass unnoticed, if not ignore completely. But this is all confined to inessential labour. We can endogenize technical progress à la Arrow, but we have seen that "the job to keep the marginal product from falling too fast as capital accumulates", that Solow had set for endogenous growth theory, turns out to be "the job of any theory where capital grows faster than the natural rate". This includes the old growth theory if we abandon the assumption of constant returns to scale, as in fact we must when we endogenize technical progress. Perhaps Solow has downplayed the role of increasing returns to scale in a theory of endogenous growth too much. True, it is an assumption which proves, as we have seen, to be neither necessary (see sections 2 and 3) nor sufficient (see sections 1 and 4-6) for endogenous growth. However, if we insist on essentiality of labour then increasing returns becomes a necessary condition for endogenous growth.

Another aspect of the comparison between exogenous and endogenous growth theory that has passed unnoticed in Solow's exposition, and our discussion has brought some light on, is strictly grammatical: the importance of the Swan Proposition is that it is an "if and only if" proposition. If we want steady states, we will have to fall within it, i.e., our production function will have to be of type (1'). That the Arrow production function does fall within it shows that the endogenization of productivity increases does not in itself have anything to do with the algebraic format of the growth model. But it is this format that is typical of the "new" school. The new growth theorists are therefore forced to leave the convention of looking for steady states. They have done it by introducing a new type of solution, asymptotic steady states. Of course in a sense all steady states are asymptotic, for if the

[15] It can be easily checked that $\beta = 1 + ANK\exp(-ANK)/[1-\exp(-ANK)]$.

[16] As well explained by Solow, it was the steady state assumption that forced economists to restrict their attention to labour-increasing technical progress. Can endogenous growth theory handle a greater variety of technical progress? I do not know.

system is not on one of them, it will (if stable) come back to it only in an infinite time. But the distinction remains. While in a model with a steady state there are initial conditions that allow the economy to be in steady state all the time, this possible constancy over time is missing in all types of endogenous growth models we have considered. Thus we can argue that the contribution of the endogenous growth theorists has been to practice a game with slightly altered, and perhaps a little more liberal, grammatical rules. The new rules facilitate the study of such obviously intellectually challenging and socially relevant problems as growth under a constant working population, and many others.

REFERENCES

[1] K. J. Arrow, "The economic implications of learning by doing", Review of Economic Studies, 29, 1962, pp. 155-173. Reprinted in J. E. Stiglitz and H. Uzawa, Readings in the modern theory of economic growth, The M.I.T. Press, 1969, pp. 210-228.

[2] G. Bertola, "Factor shares and savings in endogenous growth", American Economic Review, Vol. 83, NO. 5, December 1993, pp. 1184-1198.

[3] E. Burmeister and A.R. Dobell, Mathematical theories of economic growth, MacMillan, 1970.

[4] R. M. Solow, Siena Lectures on endogenous growth theory, Collana del dipartimento di Economia Politica, n. 6, Siena, 1992.

[5] R. M. Solow, Learning from "Learning by doing": lessons for economic growth, Stanford University Press, 1997.

[6] R.M. Solow, Growth theory: an exposition, Second Edition, Oxford University Press, 2000.

[7] T. W. Swan, "Golden ages and production functions", first published in K. E. Berrill, (ed.), Economic development with special reference to East Asia, MacMillan and St Martin's Press, 1963, and reprinted in A. Sen, editor, Growth economics, Penguin, 1970, pp.203-218.