"Bliss", Pontryagin and C.E.S

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Abstract

Authors study the problem of leisure in economic analysis, enlighted by the success of "general equilibriums stochastic models". They begin with a reappraisal of Ramsey's 1928 famous model; they show that by introduccing a CES production function, it is quite possible to have a modern de…nition of "bliss"; so Ramsey's approach is vindicated with use of modern treatment of optimal growth analysis (Pontryagin). However, since in Ramsey's model, there is no growth and since utility function has to meet special requirements when leisure is taken into account, is a reconciliation possible?

1 Introduction

Ramsey' seminal article, A Mathematical Theory of Saving, adds leisure to the conventional function of utility speci…ed ordinarily with consumption only and introduces original conception of "bliss". Strictly speaking that is not absolutely exact. In his paper Ramsey uses a two arguments utility function one is consumption, the other is labor.

The …rst part of this paper tries to reconcile Ramsey's intuitions with modern analysis. It will show that introducing Pontryagin's analysis with C.E.S production function one can de…ne …nd some de…nition of "bliss" which seems quite coherent with Ramsey's préoccupations since at the end of his paper he writes : "for simplicity let us suppose that the amount of labour is constant"

Ramsey's methodology of modelling a macroeconomic problem was too unusual for a long time, precisely until Cass (and Koopmans) wrote the fundamental paper on optimal growth theory (1965). Secondly, we observe that Cass, and many authors after him, simpli…es the function of utility of the representative agent retaining only consumption as unique argument. Work is o¤ered inelastically. And leisure has no economic value. Third, in the optimal growth theory of …rst generation, there is one state variable, capital and, maybe by symmetry, one command variable, saving that is consumption.

Nevertheless, few years after, Elizabeth Chase publishes Leisure and Consumption (1967) and Kenneth Arrow & Mordecai Kurz Public Investment, the Rate of Return and Optimal Fiscal Policy (1970) both of them using an utility function with two arguments. During the same time Sidrauski & Foley do the same thing. But there is an important difference between these three papers. In Chase's article variables are consumption and leisure, in Arrow-Kurz's it is consumption and public capital, in Sidrauski-Foley's it is consumption and money.

For convenience we study first the case of Ramsey where there is no growth and we introduce growth assumptions later. After we study new aspects of growth and cycle theories. Last we return to linear models.

1.1 A reformulation of Ramsey's approach

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Let us introduce a slight different version of Ramsey's model. Using the same utility function but introducing the discount rate $\delta > 0$ problem becomes (with contemporary usual notations)

$$\max \int_0^\infty e^{-\delta t}[U(c) - V(l)]dt$$

$$\frac{dk}{dt} = f(k;l) - c \tag{1}$$

which can be solved by Pontrjagyn's principle. The Hamilton auxiliary function is :

$$H(c;l;k;p) \equiv U(c) - V(l) + p.(f(k;l) - c) \tag{2}$$

There are two commands and one state variable; therefore two necessary conditions and one adjoint equation appear, so

$$\begin{aligned} \frac{\partial H}{\partial c} &= u(c) - p = 0 \\ \frac{\partial H}{\partial l} &= -v(l) + p.f_2(k;l) = 0 \\ \frac{\partial H}{\partial k} &= p.f_1(k;l) = -\frac{dp}{dt} + \delta.p \end{aligned} \tag{3}$$

One is back to 1928's equations[1] and also to the usual relations between consumption growth rate and the difference between marginal productivity of

[1] Combining first and second equation gives equation (2) page 546 of Ramsey.

capital and discount rate (both relations do not exist of course in Ramsey's paper). By introducing

a) an iso-elastic utility function $U(c) = (c^{1-\sigma} - 1)/(1 - \sigma)$[2],

b) a desutility function $V(l) = (l^{1-\gamma} - 1)/(1 - \gamma)$ with $\sigma > 0$; $\gamma < 0$ and

c) a Cobb-Douglas production function $f(k; l) = A.k^{\alpha}.l^{1-\alpha}$,

one determines a stationay state defined by:

$$\begin{aligned} \alpha.A.k^{\alpha-1}l^{1-\alpha} - \delta &= 0 \\ c^{-\sigma}.(1-\alpha)A.k^{\alpha}.l^{-\alpha} - l^{-\gamma} &= 0 \\ A.k^{\alpha}.l^{1-\alpha} - c &= 0 \end{aligned} \tag{4}$$

With such formulation, this problem has no acceptable solution when $\delta = 0$. First of the above equations comes out to $l/k = 0$ and consequently either $l = 0$ or $k \to \infty$. If $l = 0$ then we get $c = 0$ with imposibility for the second equation; if $k \to \infty$ then $c \to \infty$ and some difficulties (at least...) for second equation and difficulties of economic interpretation. With $\delta \neq 0$ one computes a solution, providing the jacobian is non-zero, for instance with $\sigma = 1$ one gets out $l^* = (1 - \alpha)^{1/(1-\gamma)}$; $k^* = l^*.(\delta/\alpha A)^{1/(\alpha-1)}$; $c^* = A.k^{*\alpha}.l^{*1-\alpha}$ notice that with $0 < \alpha < 1$ and $\gamma < 0$ then $0 < l^* < 1$.

What happens if $k_0 \neq k^*$ and $l_0 \neq l^*$? .Consequently, the differential system writes as:

$$\frac{dk}{dt} = A.k^{\alpha}.l^{1-\alpha} - c \tag{5}$$

$$\frac{dc}{dt} = \frac{c}{\sigma}\left(\alpha.A.k^{\alpha-1}l^{1-\alpha} - \delta\right) \tag{6}$$

$$(l^{-\gamma-1} - c^{-\sigma}\alpha(1-\alpha)Ak^{\alpha}l^{-\alpha-1})\frac{dl}{dt} = \tag{7}$$

$$c^{-\sigma}(1-\alpha)Ak^{\alpha}l^{-\alpha}\left\{\delta - \alpha Ak^{\alpha-1}l^{1-\alpha} + \frac{\alpha}{k}(Ak^{\alpha}l^{1-\alpha} - c)\right\} \tag{8}$$

It is quite unconvenient even with simple specifications and always takes us to a saddle point solution. The assumption $l = l^*$ with $k \neq k^*$ is not so general but much more usable. Jacobian comes out to be:

$$J = \begin{pmatrix} \delta & -1 \\ \frac{c^*}{\sigma} f_{11}(k^*; l^*) & 0 \end{pmatrix} \tag{9}$$

and it is well-known that there is two distinct eigenvalues one positive and one negative and so the stationnary equilibrium is a saddle point. Except, of course, when $\delta = 0$ which appears, again, as impossible.

[2] Page 549 Ramsey uses the function

$$U(x) = \frac{x^2}{15000} + \frac{13.x}{300} - 3$$

advising it's an approximation of the actual function.

The above form of the net utility function is here acceptable since there is no growth. As a matter of fact, the stationary solution is obviously constant.

1.2 Assumption of CES function

What happens with CES production function ? Let us make the assumption $f(k;l) = A.(\alpha.k^{-\rho} + (1-\alpha).l^{-\rho})^{-1/\rho}$ where $\rho + 1$ is inverse of elasticity factor substittution (with of course $\rho > -1$); when $\rho \to 0$ such expression can be transformed in a Cobb-Douglas production fonction $A.k^{\alpha}.l^{1-\alpha}$. We are specially interested in case $\rho > 0$, where elasticity factor substitution $1/(1+\rho)$ is less than 1. In such a case, $k \to \infty$; $f(k;l) \to A.(1-\alpha)^{-1/\rho}.l$; then there is appearance of an horizontal asymptote; its height is a linear function of l;with $l > 0$. Consequently, when k is quite large and for stationary state $c = f(k;l)$, relation between c et l is a linear one.

If one wants to get more consumption, it is necessary to have larger work supplies (and of course less work supply for less consumption). Choice about l depends only on utility function. Now, Ramsey makes the (quite usual) assumption of an non-increasing marginal utility for consumption and non marginal decreasing marginal utility for work. As a mater of fact, and for simplicity, marginal utility for consumption is strictly decreasing $U'(c) = u(c) = c^{-\sigma}; \sigma > 0$, and marginal utility for work is increasing $V'(l) = v(l) = l^{-\gamma}; \gamma < 0$.

Consequently, instant choice is given by solving $\max_{c;l}(U(c) - V(l))$ i.e, (by taking into account our observation concerning introduction of a CES production function) , $\max_l(U(A.(1-\alpha)^{-1/\rho}.l) - V(l))$; its solution is given by the necessary condition:

$$A^{-\sigma}.(1-\alpha)^{\sigma/\rho}.l^{-\sigma} - l^{-\gamma} = 0 \tag{10}$$

Such condition is also sufficient since second derivative is negative; and solution is then:

$$l = A^{\sigma/(\sigma-\gamma)}.(1-\alpha)^{\sigma/\rho(\sigma-\gamma)} \tag{11}$$

Such quantity is of course a positive one; when introduced in the net utility function, it gives us the amount of B, "bliss" which does not depend of the value of the discount rate δ which may even be equal to zero.

These developments allow to imagine, and may be to vindicate, Ramsey's analysis who might have imagined a production function with low elasticitity of substitution, may be even, as suggested in the above citation, with complementary factors.

2 New directions

In the second part of his paper, Ramsey assumes

$$f(k,l) = r.k + w.l \tag{12}$$

with r and w independant and both constant. So, $f_{11} \equiv 0$ and eigenvalues are 0 and $\delta \neq 0$. A very interesting feature of that assumption, not frequently enlighted before, yet obvious, is growth never cease in such a case. Ramsey's assumption is Romer's assumption ! In footnote 1 page 549 author precises "...we only require independance and not constancy, and that nowhere do we really require wages to be constant,...". For large values of k with CES production function $f_1 \approx 0$ and $f_2 \approx A.(1-\alpha)^{-1/\rho}$ remains constant when l varies.

The problem is written as

$$\max_{c,l} \int_0^{\infty} e^{-\delta t}[U(c) - V(l)].dt \tag{13}$$

$$\frac{dk}{dt} = r.k + w.l - c \tag{14}$$

$$k(0) = k_0 \quad l(0) = l_0$$

Using as before Maximum Principle leads to the well-known relation

$$\frac{c.u'(c)}{u(c)}\frac{1}{c}\frac{dc}{dt} = \delta - r \tag{15}$$

therefore if $\delta < r$ then $c \to +\infty$ else $c \to 0$. The second necessary condition

$$-v(l) + u(c).w = 0 \tag{16}$$

corresponds to equation (2) in Ramsey's and the not numbered equation in front of page 550. If $c \to +\infty$ then $u(c) \to 0$ and $l \to 0$. The whole production comes from capital which growths without limit and workers do nothing.

Ramsey treats that point with particular attention. First, with the same equation he observes that l is a function of c and, Second, with $y = c - w.l$ he defines as unearned income (but is simply the difference between consumption and wages) and defines a new utility function ω so $\omega(y) = u(c) = v(l)/w$ and $\Omega(y) = \int \omega(y).dy$. In fact, as seen before, with the assumption $l = \phi(c)$ utility depends on one variable and k and δ become unimportant. Nevertheless with $r > 0$ we do think there are a lot of unresolved problems !

2.1 Rate of discounting and endogenous growth ?

It seems quite interesting to examine more thoroughly last part of Ramsey's paper since he does introduce rate of discounting: when considering "the problem of the determination of the rate of interest", he supposes "in the first place" "that everyone discounts future utility for himself or his heirs, at the same rate ρ". Then "in a state of equilibrium there will be no saving and ...three equations ..". "The last equation tells us that the rate of interest as determined by the marginal productivity of capital $\partial f/\partial c$; must be equal to the rate of discounting ρ"[3]:

"But suppose, continues Ramsey, that at a given time, say the present $\partial f/\partial c > \rho$: Then, there will not be equilibrium, but saving, and since a great deal cannot be saved in a short time, it may be centuries before equilibrium is reached, or it may never be reached, but only approached asymptotically; and the question arises as to how, in the mean time, the rate of interest is determined, since it cannot be by the ordinary equilibrium equation of supply and demand".

Some interesting questions are already present in the above citation. When Ramsey contemplates the possibility of asymptotically approaches to equilibrium, it seems quite obvious that there is some quite of convergence towards long term equilibrium - which is quite similar to the different approaches of "exogenous growth theories" (Solow, or Cass-Koopmans). And, when arises the interrogation "as to how, in the mean time, the rate of interest is determined", one comes to some vision of "endogenous growth theory".

In the different versions of "endogenous growth theories", we have chosen to deal with "two sectors" models for two reasons; first, it deals with allocation of work between two sectors producing physical capital and composite good for the first one and human capital for the other; second, the determination of the rate of interest is quite clear here and quite interesting comparisons can be made.

3 About modern growth theories

3.1 Introducing positive growth rate

If growth rate is strictly positive, utility function must be strictly specified. That is a problem which Solow met in his objections to endogeneous growth approach (cf his Siena Conferences) but which is not specific to "endogeneous" approaches.

In contemporary analysis, (general equilibriums stochastic models), the

[3] Ramsey's notation has been conserved here

representative agent disposes one unit of time, or ¹ units, ¹ < 1. Leisure is the di¤erence between endowment and working time. The endowment is stationary and so is working time and leisure unlike consumption which can growths continuously. therefore the two arguments of the utility function behave very di¤erently and the form of the function cannot be freely chosen, at least if growth rate is positive.(see King, Plosser & Rebelo 1988, Barro & Sala-i-Martin 1995, Abraham-Frois & Goergen 1996 e.g.). For instance,

$$u(c_t; l_t) \text{ ´ } \frac{[c_t.V(l_t)]^{1 \text{ ¡ ¾}} \text{ ¡ } 1}{1 \text{ ¡ ¾}}; \qquad V(l_t) = \exp(\text{ ¡ } \frac{l_t^{1+x}}{1+\text{Â}}) \tag{17}$$

(cf. Farmer "Macroeconomics of Self-ful…lling prophecies", where ¾ is inverse of time substitution elasticity of consumption and Â º 0).

So the problem we meet here is to try a reconciliation between our treatment of Ramsey's bliss and the particular speci…cation for utility function which appears as necessary when one introduces growth assumption ? Our conjecture is that is normally impossible; the only possibility of reconciliation would be a treatment by "intensive" units as P.Massé did in his noticeable contribution of 1969 "Croissance optimale, théorèmes limites, particularisations". Technical progress enlarges possibilities of consumption but, simultaneously, makes heavier costs. We have to di¤erentiate labor, which is more e¢cient and working time, or leisure, which does not grow at a positive rate. But, now, the model is very far from Ramsey's.

3.1.1 Back to Rebelo's model

Let us come …rst to some notations and reminds. We shall deal …rst with Rebelo's model with two production sectors and production functions such as:

$$Y = C + \dot{K} + \text{µ}K = A(vK)^{\text{®}}.(uH)^{1 \text{ ¡ ®}} \tag{18}$$

$$\dot{H} + \text{µ}H = B.[(1 \text{ ¡ } v).K]^{\text{´}}.[(1 \text{ ¡ } u).H]^{1 \text{ ¡ ´}} \tag{19}$$

In such formulation, there are two commodities both produced with Cobb-Douglas production functions and constant returns to scale, and different exponents ® and ´:

Y is the amount of "physical" (composite) good - which can be used as well for consumption or investment -; so, here , we are always in "corn-model" world; but, there comes also a production function for human capital H. A and B are both positive constant parameters,® and ´, both between 0 and 1, represent the rate of capital remuneration in both sectors; v and u are the fractions of physical total capital and total human capital used in each good's production. Physical capital depreciates at constant rate µ. We use the simplifying assumption that the same depreciation rate is used for human capital.

The usual assumption is made along which education sector is relatively more intensive in human capital utilization that physical good sector, i.e.: ´ < ® (which seems relatively plausible); in a symmetric way, production of physical good is relatively intensive in physical capital use.

An important feature is the whole disappearance of any reference to non quali…ed work; this primary factor is however always present but only as "human capital support", or "vector". But growth process will entail increase in quali…cation, with constant amount of "simple" work available.

3.1.2 Determination of r

It is quite clear that in the two preceding equations, one has constant returns to scale relatively to the two accumulable factors K and H. When taking into account steady states of growth, v and u are constant and all global macro-economic variables i.e. C; K; H et Y grow at the same rate. Marginal productivity of physical capital in "goods" sector, and consequently rate of interest comes to:[4]:

$$r = A\text{®}(vK)^{\text{®} ¡ 1}:(uH)^{1 ¡ \text{®}} = A\text{®}(\frac{vK}{uH})^{¡(1 ¡ \text{®})} \quad (20)$$

In steady state of growth, (K=H) ratio stays constant; so there will be possibility of constant rate of growth; this comes out because one has two accumulable factors[5] and that they can grow at the same rate. One gets quite easily the consumption growth rate from the usual formulation: $\text{°} = \text{¾}(r ¡ \text{µ} ¡ \text{±}) = \text{°} = \text{¾}(A\text{®}(\frac{vK}{uH})^{1 ¡ \text{®}} ¡ \text{µ} ¡ \text{±})$:

3.1.3 Steady-sate and inter-sectorial inputs allocation

Physical capital as well as human capital get same return in both sectors. After some calculations and rearrangements, on comes to the following relation between v and u:

$$(\frac{\text{´}}{1 ¡ \text{´}}):(\frac{v}{1 ¡ v}) = (\frac{\text{®}}{1 ¡ \text{®}}):(\frac{u}{1 ¡ u}) \quad (21)$$

Consequently, there is a positive relation between v and u; simple cases are v = 1 for u = 1 and v = 0 for u = 0: And, for given values of ® and ´ [6], an increase in amount produced of goods (consumption or investment) can

[4]It must be clear that the amount of capital in this sector is not K but vK, which explains the value of the derivative.

[5]here is the important di¤erence with the usual models where the intresting ratio was K/L, with L as a primary non accumulable factor

[6]Quite obviously ® = ´ implies u = v(same technique)

be realized by simultaneous increase in amount of both factors, K and H to both production sectors.

So, one gets same growth rate for $Y; C; K; H$ in steady-sate. Values of α and β fix the relation between u and v; and allocation of the two kinds of capitals between the two production sectors.

If one makes the assumption of same rate of return for both capital goods (physical and human capital) in steady-state, it is possible to show that this rate r^* is a constant ; its value depends on production functions parameters; then, one can obtain quite easily growth rate for all macro-economic variables.

3.1.4 About steady-state stability

Let us note p the ratio of marginal productivity of human capital in first sector to marginal productivity of human capital in the second sector; it appears that this ratio is only function of vK/uH: An interesting result about steady-state stability appears after rather heavy calculation[7]. If $\gamma_p = \frac{1}{p}\frac{dp}{dt}$ is the relative rate of change of this ratio, there appears a differential equation with stability (i.e.e $\partial\gamma_p/dp < 0$ if $\alpha > \beta$) and instability in the opposite case. Consequently, in the case generally considered as plausible $\alpha > \beta$; so, there is steady convergence of p towards its steady- state-value. It can be inferred that there is also convergence of vK/uH to its value on steady-state, which settles consequently marginal productivity of physical capital in the goods sector; the latter will be constant and so one comes to stability of consumption growth rate and all macro-economic growth rates

3.2 What about cycles and leisure ?

Treatment with OLG seems relevant. With the representative agent hypothesis some models put a maximum value for the time to work. All time not working is leisure. Either the agent chooses to offer some quantity of labor or to enjoy farniente. In OLG models the choice is given by optimization of a Ramsey-like utility function. In such models there is no growth, consumption and leisure have same order of value and so their utility. The aim of that models is to explain cycles or self-fulfilling prophecies (Azariadis & Guesnerie 1982, Farmer 1987, Reichlin 1987).

Cycles are solutions of deterministic models. Generally a stationary equilibrium exists and becomes unstable when a parameter goes on, in as such as a cycle, by bifurcation, appears. Self-fulfilling prophecies are solutions of stochastic models. We just take a glance to Reichlin's model.

Among various interesting models we choose to enlight some features of

[7]Cf. R.Barro et X.Sala-i-Martin, Theory of economic Growth. Ch. 5

contemporary modellization by looking at the model presented by Reichlin[8] in Nonlinear Economics Dynamics, Grandmont ed. 1987.

Endogenous equilibrium cycles exist in Grandmont On endogenous competitive business cycles, Econometrica, 1985. They are driven by flip bifurcation, but flip bifurcation are not very robusts and in Grandmont's model saving must be a decreasing function of the interest rate. The main feature of Reichlin's model is the possibility of cycles in the neighbourhood of the steady state by Hopf bifurcation.

Representative household lives two periods. In the first he works offering l units of labour, in the second he consumes c he chooses l and c maximizing $u(c) - v(l)$ under budget constraints with $c; l > 0$. The technology is described by the production function

$$x = \min\{\frac{l}{a_0}; \frac{l}{a_1}\} \tag{22}$$

where x is the amount of corn produced, l is the stock of seeds left over the previous production period after consumption $l_t = x_{t-1} - c_{t-1}$. Budget constraints are obvious : $l_{t+1} \leq w_t.l_t; c_{t+1} \leq l_{t+1}.R_{t+1}$ where R is the real interest factor and w the real wage rate. It is assumed that $u'(c) > 0; u''(c) < 0; v'(l) > 0; v''(l) > 0$ and $\lim_{l \to \infty} v'(l) = +\infty; \lim_{l \to 0} v'(l) = 0$. With these assumptions (and $0 < a_1 < 1$) the problem has a unique solution satisfying $l; c > 0$.

Now if we assume that the demand for leisure and consumption satisfies gross substitute axiom : $c.u''(c) + u'(c) > 0; \lim_{c \to 0} c.u'(c) = 0; \lim_{c \to \infty} c.u'(c) = +\infty$, and the technical assumptions asserting that eigenvalues of the jacobian cross the unit circle : the elasticity of u' is strictly under one in absolute value and the elasticity of v' is finite, then it exists an invariant attractive circle in a left neighborhood of a value a_1^* of a_1.

As Reichlin says : "The origin of this dynamic behavior is to be found in the opposing effects on saving of wage income and intertemporal substitution caused by factor price movements. For this two effects to generate the cycle, a sufficiently low elasticity of substitution between capital and labor is required".

4 From novelty to tradition: back to linear models of production ?

"New" models dealing with "endogenous growth theory" differ on many points from "traditional views" 'a la' Ramsey or Solow. The main difference

[8]Pietro Reichlin Equilibrium Cycles in an Overlapping Generations Economy with Production.

concern may disappearance of traditional treatment of labor with production functions such as Y = F(K,L); more precisely, non quali…ed labour is only a vector, a support for human capital, for quali…cation when dealing with two production sectors models such as Y = F(K,H). There is no more speci…c payment for labor; the return for human capital is just as return for physical capital (of course under the traditional assumption of equal returns for di¤erent sorts of capital inputs). On the other hand, traditional limits to growth, founded on labor growth rate and possibility of productivity growth are no more relevant. One is back to Von Neumann approach, production of robots by means of robots

Some precisions are quite necessary to facilitate the comparison: linear models of production are founded on …xed coe¢cients production functions when in endogenous growth theory models founded on physical/ human capital models we generally …nd substitutability (Cobb-Douglas) production functions. However, two observations must be made; …rst, in the Lucas-Uzawa case, there is linearity in human capital production; second, and more important, in the general case (Rebelo approach), steady state main features are determined by …a Jacobian matrix (this matrix is of course reducible in the Lucas-Uzawa model).

@F=@K @F=@H

@H=@K @H=@H

The Jacobian characteristics settle capital marginal productivity (in Lucas' model, human capital production conditions are …xed and human capital is the only "basic" commodity according to Sra¤a's de…nitions); quite obviously, this is just the maximum eigenvalue of the Jacobian matrix.

It is well known that in linear production models maximum rate of pro…t R and maximum rate of growth G are simultaneously determined (if the matrix is an irreducible one) by equation systems p = (1 + R)Ap or ®p = Ap and y = (1 + G)yA or ®y = yA with ® = 1=(1 + R) = 1=(1 + G):

Obviously, these features are not di¤erent from those appearing in endogenous growth two-sectors models. However, it must be quite clear that in these models, value of steady-growth rate is determined by di¤erence between capital rate of return and discounting rate when this latter is considered as zero in linear production models, since they just deal with comparison between steady states. And of course, we are again back to Ramsey's position: "it is assumed that we do not discount later enjoyment in comparison with earlier ones, a practice which is ethically indefensible and arises merely from the weakness of the imagination"

It could be interesting to make further comparisons; it is well known that in "linear" production two-sectors models (let us say corn-iron), one has di¤erent possibilities concerning for instance (w; r) convexity or concavity, Wicksell e¤ects, "transformation" problems. It is well known that the key is the difference in capital-intensities (k_i/l_i) between the two sectors.Uzawa-Lucas model structure is quite similar to two-sectors iron-corn model where iron is necessary for corn production, but corn is not necessary for iron production; human capital is necessary for "goods" production but "goods" are not necessary for human capital. Uzawa-Lucas model is reducible although Rebelo's general model matrix is irreducible. The interesting point is that stability is quite different according to the value of (K_i/H_i) ratio in the different sectors...

4.0.1 A note on Lucas-Uzawa's model[9] [10]

There appears a huge simplification compared to previous model since the assumption is $\eta = 0$; which means that human capital is produced without any physical capital (neither buildings nor buildings). So production functions come to:

$$Y = C + \dot{K} + \theta K = AK^{\alpha}.(uH)^{1-\alpha} \tag{23}$$

$$\dot{H} + \theta H = B.[(1-u).H] \tag{24}$$

Moreover, Lucas makes the assumption of equal rate of return for both kinds of capital; Since, rate of return of human capital in human capital sector is $\partial \dot{H}/\partial(1-u)H = B = cste$; one gets: $F'(K) + \theta = B$.

With usual assumptions, consumption growth rates is now: $\gamma = \sigma(B - \theta - \delta)$.

5 Conclusion

As soon as one complexifies an economic model, a lot of constraints appears either on the form of the production function or on that of utility function. Of course, some of them come up with the aim of the work. And it is one aspect of improvement of economics to recognize these constraints.

Moreover, the questions put at economists are always almost the same. Only context varies. And so fashions.

We may recognize a requirement of preciseness and the concern of adequation to the so called stylized facts in a more elaborate mathematical pattern. Old growth theories, like Morgenstern-von Neumann's, was concerned solely with the equilibrium growth path but they try to deal with a multi-sectorial economy.

[9] Lucas, R.E. (1988): "On the Mechanics of Economic Development" Journal of Monetary Economics, july

[10] Uzawa, H. (1965): "Optimal Technical Change in a Model of Economic Growth" International Economic Review, january

Current theories do not try to explain, or to deal with, the multiplicity of goods, they break o¤ with the old form of multi-sectoriality. They use variables not directly observable and frequently perform a one physical good model. Nevertheless they do come out into sight transitory paths. And, maybe the most important, old growth theories were linear, news are not.